System Considerations of Pre-filtering and Post-filtering

System Considerations of Pre-filtering & Post-filtering

- Introduction
- Spectra of Sampled Signals
- Prefiltering
- Postfiltering
- Oversampling Approach
- Conventional Switched-Capacitor Approach
- Examples

Filters

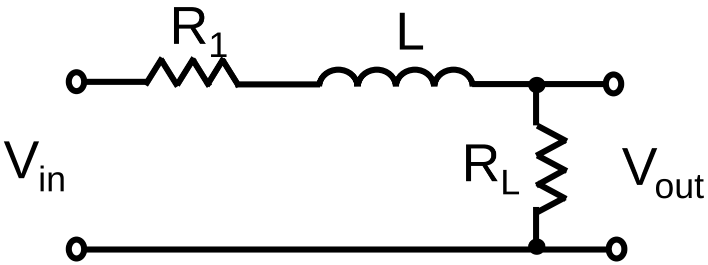
- Continuous-time filter
  - RLC passive
  - Active RC

- Sampled-Data filter
  - Switched-Capacitor filter

- Digital filter

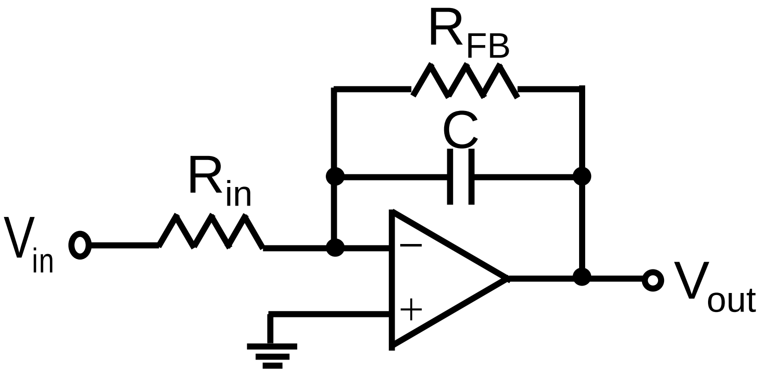
Continuous-Time Filters

- Example: 1 pole low pass filter
  - ◆ Passive



$$\frac{V_{out}}{V_{in}} = \frac{R_L}{R_1 + R_L}\left[\frac{1}{1+\dfrac{sL}{R_1+R_L}}\right]$$

  - ◆ Active



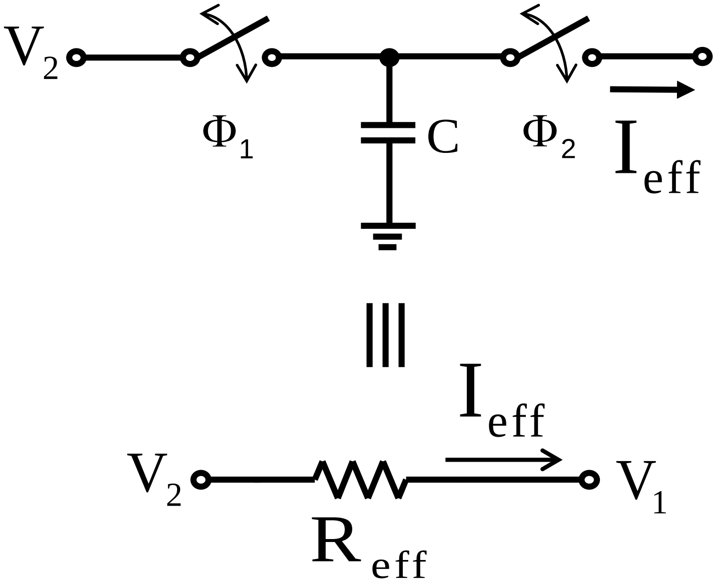
$$\frac{V_{out}}{V_{in}} = -\frac{R_{FB}}{R_{in}}\left[\frac{1}{1+sR_{FB}C}\right]$$

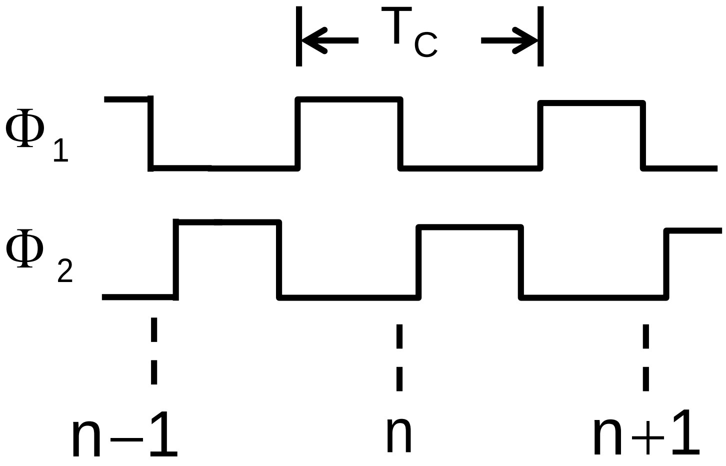
  - ◆ Equivalence conditions of the above

$$R_{in} = R_1 \, , R_{FB} = \frac{R_1R_L}{R_1 + R_L} \, , C = \frac{L}{R_1R_L}$$

Switched-Capacitor Filter (SCF)

- Basic concept

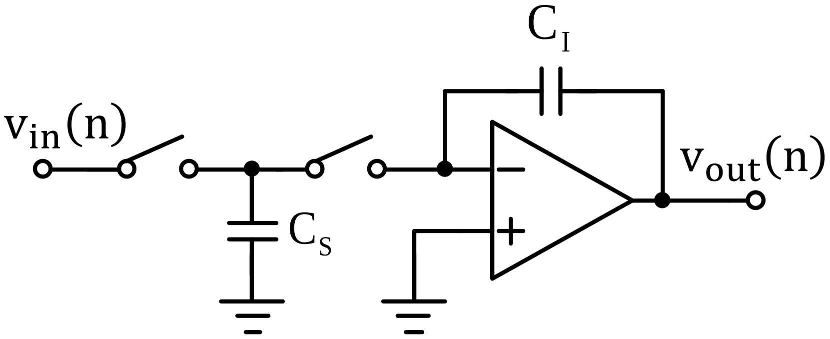




$$\Delta Q = C(V_2 - V_1) \;\; => \;\; I_{eff} = \frac{\Delta Q}{T} = \frac{V_2 - V_1}{R_{eff}} \;\; => \;\; R_{eff} = \frac{T_C}{C}$$

SCF (Cont.)

- Example: SC integrator (stray-sensitive)



$$v_{out}(n) - v_{out}(n-1) = -\frac{C_S}{C_I} v_{in}(n-1)$$

$$\Rightarrow v_{out}(n) = v_{out}(n-1) - \frac{C_S}{C_I} v_{in}(n-1)$$

$$\Rightarrow V_{out}(z) = z^{-1} V_{out}(z) - \frac{C_S}{C_I} z^{-1} V_{in}(z)$$

$$\Rightarrow H(z) = \frac{V_{out}(z)}{V_{in}(z)} = -\frac{C_S}{C_I} \frac{z^{-1}}{1-z^{-1}}, \text{ where } z = e^{j\omega T}$$

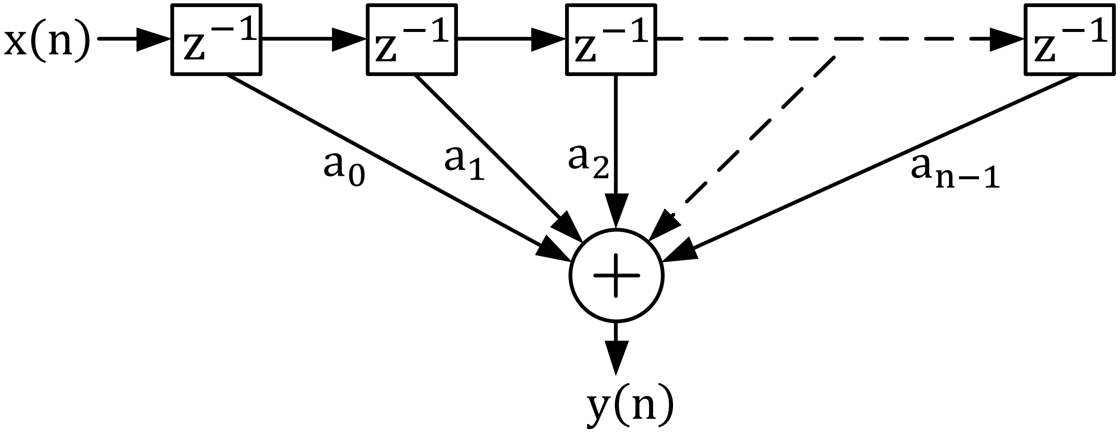
If ωT<<1

$$\lim_{\omega T << 1} H\left[e^{j\omega T}\right] = -\frac{C_S}{C_I} \frac{e^{-j\omega T}}{1-e^{-j\omega T}} = -\frac{C_S}{C_I} \frac{\left[1 - j\omega T + \frac{(j\omega T)^2}{2} - \cdots\cdots\right]}{j\omega T - \frac{(j\omega T)^2}{2} + \cdots\cdots}$$

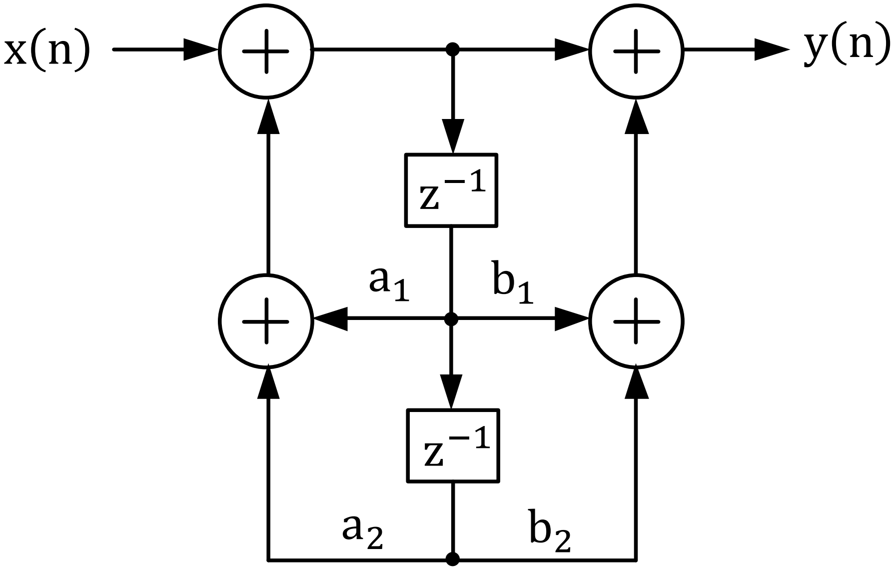
$$\approx -\frac{C_S}{C_I} \frac{1}{j\omega T} = -\frac{1}{j\omega\left(\frac{T}{C_S}\right) C_I} = -\frac{1}{j\omega R_{eff} C_I}$$

Digital Filter

- FIR (Finite Impulse Response)

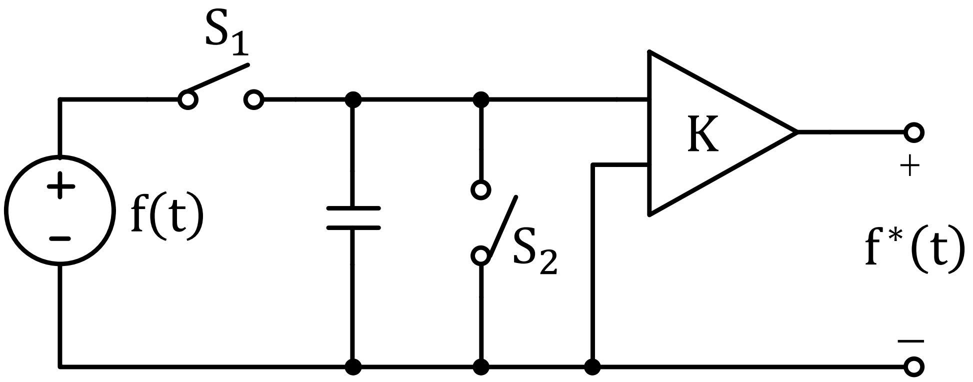


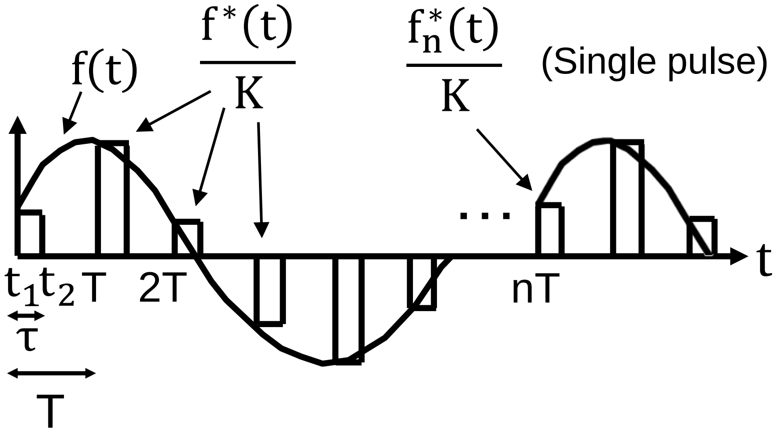
- IIR (Infinite Impulse Response)

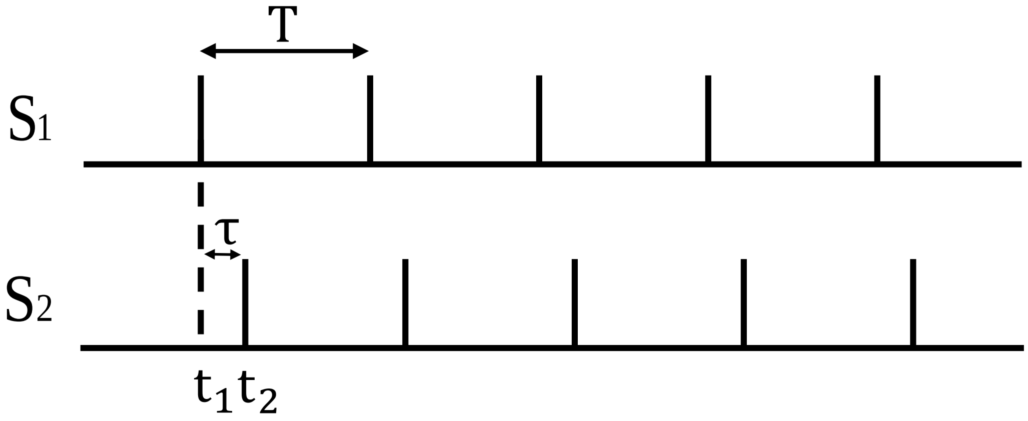


- Operations of digital filter
  - ◆ Multiply
  - ◆ Delay
  - ◆ Add

Analog Sampling Circuit



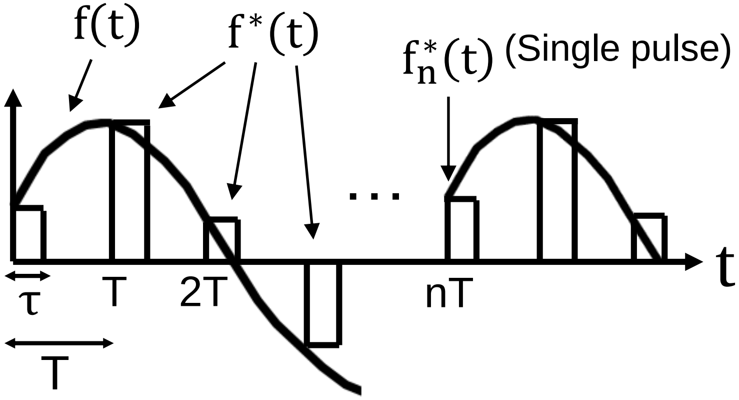




- Original continuous-time signal $f(t)$
- Sampled/Held Signal $f^*(t)$

Analog Sampling Circuit (Cont.)

- Mathematical relationship (assume K=1)
  - Single-pulse signal $f_n^*(t)$



$$\rightarrow f_n^*(t) = \frac{f(nT)}{\tau}[\theta(t-nT) - \theta(t-nT-\tau)]$$

$$\text{where } \theta(t) \equiv \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases} \qquad (13.3)$$

  - Sampled signal for all time

$$\rightarrow f^*(t) = \sum_{n=-\infty}^{\infty} f_n^*(t) = \sum_{n=-\infty}^{\infty} \frac{f(nT)}{\tau}[\theta(t-nT) - \theta(t-nT-\tau)] \qquad (13.4)$$

  - Laplace transform for $f_n^*(t)$ and $f^*(t)$

$$\rightarrow F_n^*(s) = \frac{1}{\tau}\left(\frac{1-e^{-s\tau}}{s}\right) f(nT)e^{-snT} \qquad (13.5)$$

$$\rightarrow F^*(s) = \frac{1}{\tau}\left(\frac{1-e^{-s\tau}}{s}\right) \sum_{n=-\infty}^{\infty} f(nT)e^{-snT} \overset{(\text{for } \tau \rightarrow 0)}{=} \sum_{n=-\infty}^{\infty} f(nT)e^{-snT} \qquad (13.7)$$

Signal Spectra of Zero-Width Samples

- For $\tau \to 0$ (Zero-width sampling)

$$f(t) \longrightarrow F(j\omega) \quad \cdots\cdots\cdots\cdots\cdots \quad \text{Original signal}$$

$$f^*(t) \longrightarrow F^*(j\omega) \quad \cdots\cdots\cdots\cdots\cdots \quad \text{Sampled-data signal}$$

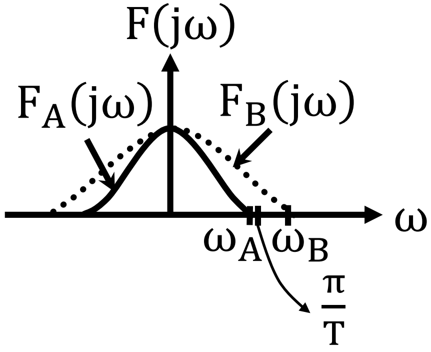
- Spectrum of sampled signal
  - ◆ Method1: Replace **s → jω** in (13.7)
  - ◆ Method2:

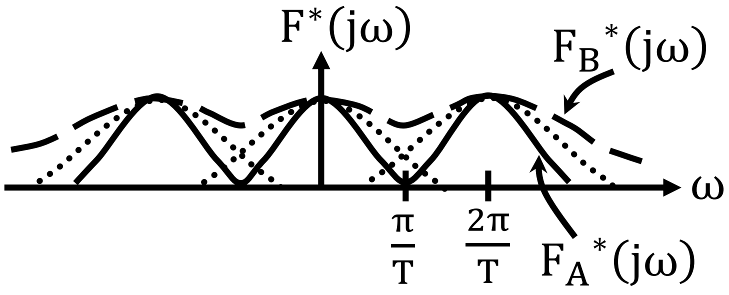
$$f^*(t) = f(t)\cdot s(t) \xrightarrow{\text{Fourier transform}} F^*(j\omega) = \frac{1}{2\pi} F(j\omega) \overset{\substack{\text{Convolution} \\ \downarrow}}{\otimes} S(j\omega)$$

$$\text{where } s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$S(j\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$\Rightarrow F^*(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(j\omega - jk\frac{2\pi}{T}\right)$$





- Continuous-time signals $f_A(t)$ and $f_B(t)$
- Sampled-Data signals $f_A^*(t)$ and $f_B^*(t)$

Nyquist Theorem

- There is a one-to-one relation between values

  $F_{A/B}(j\omega)$ and $F^*_{A/B}(j\omega)$
  - ◆ Replicas forming $F^*_B(j\omega)$ overlap
    - ➢ It is called aliasing or folding and results in nonlinear distortions

- Low-pass Filter

$$F_A{}^*(j\omega)H(j\omega) = F_A(j\omega) \text{ where } H(j\omega) = \begin{cases} 1 \ ; & \left|\omega \le \frac{\pi}{T}\right| \\ 0 \ ; & \left|\omega > \frac{\pi}{T}\right| \end{cases}$$

  - ◆ The continuous-time signal $f_A(t)$ is recovered
  - ◆ But no such operation can regain $F_B(j\omega)$ from $F_B{}^*(j\omega)$

- Nyquist first observes this phenomenon
  - ◆ Nyquist Theorem : $f_{sampling} > 2f_{signal\ band}$

Sample/Hold (S/H) Effect

- Due to nonzero-width samples, assume $\tau = \mathrm{T}$ and from (13.5)

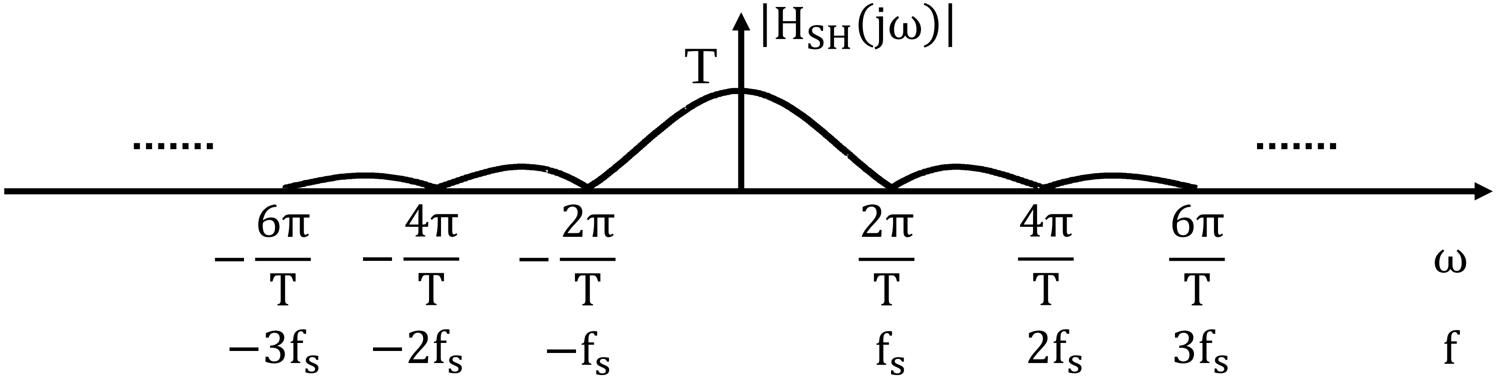
→ $\mathrm{F_{SH}(j\omega) = H_{SH}(j\omega)F^*(j\omega)}$, where $\mathrm{F^*(j\omega)}$ is spectra of zero-width samples

- Spectrum of S/H response

$$\mathrm{H_{SH}}(s) = \frac{1-\mathrm{e}^{-\mathrm{sT}}}{\mathrm{s}}$$

Let s → jω

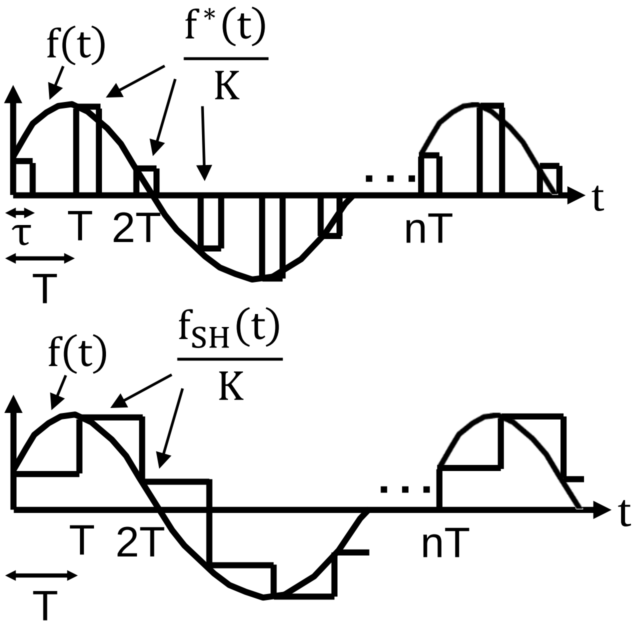
$$\mathrm{H_{SH}(j\omega)} = \frac{1-\mathrm{e}^{-\mathrm{j\omega T}}}{\mathrm{j\omega}} = \mathrm{Te}^{-\frac{\mathrm{j\omega T}}{2}}\left(\frac{\sin\frac{\omega\mathrm{T}}{2}}{\frac{\omega\mathrm{T}}{2}}\right) \text{ or } |\mathrm{H_{SH}(f)}| = \mathrm{T}\frac{|\sin(\frac{\pi\mathrm{f}}{\mathrm{f_s}})|}{|\frac{\pi\mathrm{f}}{\mathrm{f_s}}|}$$

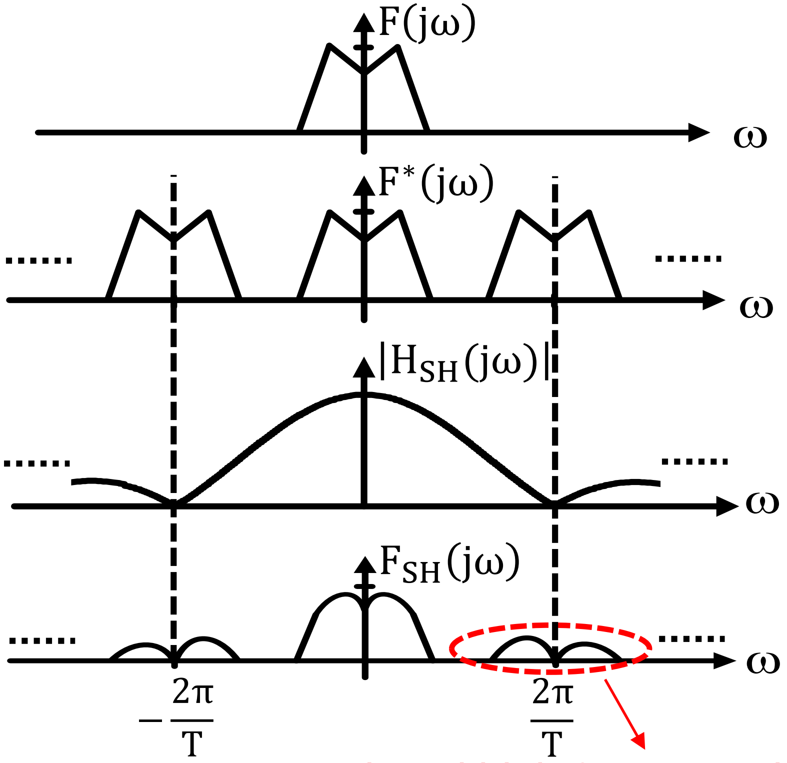


- $\mathrm{H_{SH}(j\omega)}$ has a linear phase. Its amplitude has sinc response (sinX/X)

Sample/Hold (S/H) Effect (Cont.)

- $F_{SH}(j\omega) = H_{SH}(j\omega)F^*(j\omega)$
- $F^*(j\omega)$ is formed by replicas of $F(j\omega)$
- $F_{SH}(j\omega)$ is replicated and multiplied by the sinX/X response.



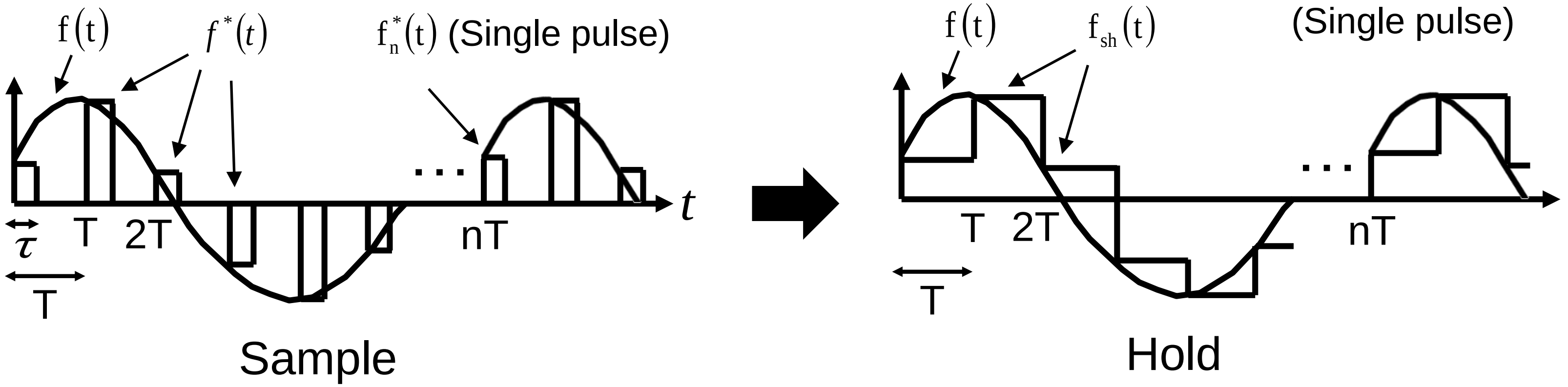


Reduced high-frequency images

- S/H before an ADC
    - ◆ Allow ADC to have a constant input value during one conversion
    - ◆ Relax the anti-aliasing requirement

Derivation of S/H Function H_{SH} (s)

- S/H response (refer to p.9-8)



- ◆ For sampled signal: $f^*(t) = \sum_{n=-\infty}^{\infty} \frac{f(nT)}{\tau}\left[\vartheta(t-nT) - \vartheta(t-nT-\tau)\right]$ (13.4)
- ◆ Assume τ = T (Nonzero-width sampling)

→ $f_{sh}(t) = \sum_{n=-\infty}^{\infty} f(nT)\left[\vartheta(t-nT) - \vartheta(t-nT-T)\right]$ (13.39)

- ◆ Laplace transform: $F_{SH}(s) = \frac{1-e^{-sT}}{s} \sum_{n=-\infty}^{\infty} f(nT)e^{-snT} = \frac{1-e^{-sT}}{s} F^*(s)$ (13.40)

→ Sample/Hold transfer function, H_{SH} (s), is equal to

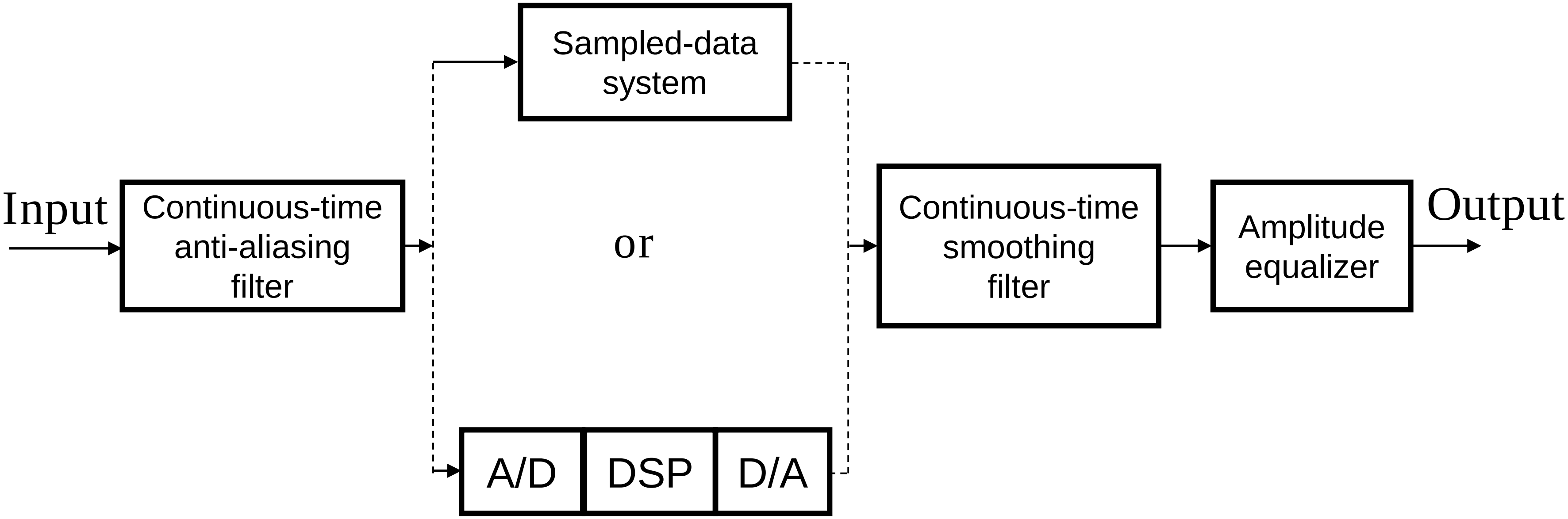
$$H_{SH}(s) = \frac{1-e^{-sT}}{s} \quad (13.41)$$

Sampled-Data System with Continuous-time Input & Output Signals

- Distortion due to $H_{SH}(j\omega)$ is linear as opposed to nonlinear aliasing distortion

- $F(j\omega)$ can be recovered from $F_{SH}(j\omega)$ by two steps
  - Low-pass filter
  - Amplitude equalizer with a transfer function

$$H_{EQ}(j\omega) = {}^{1}\!\!\diagup_{H_{SH}(j\omega)}$$



- The anti-aliasing and smoothing filters can be identical lowpass filters, and should ideally have sharp cutoff frequency (except oversampling rate signal processing where Decimation and interpolation are used)

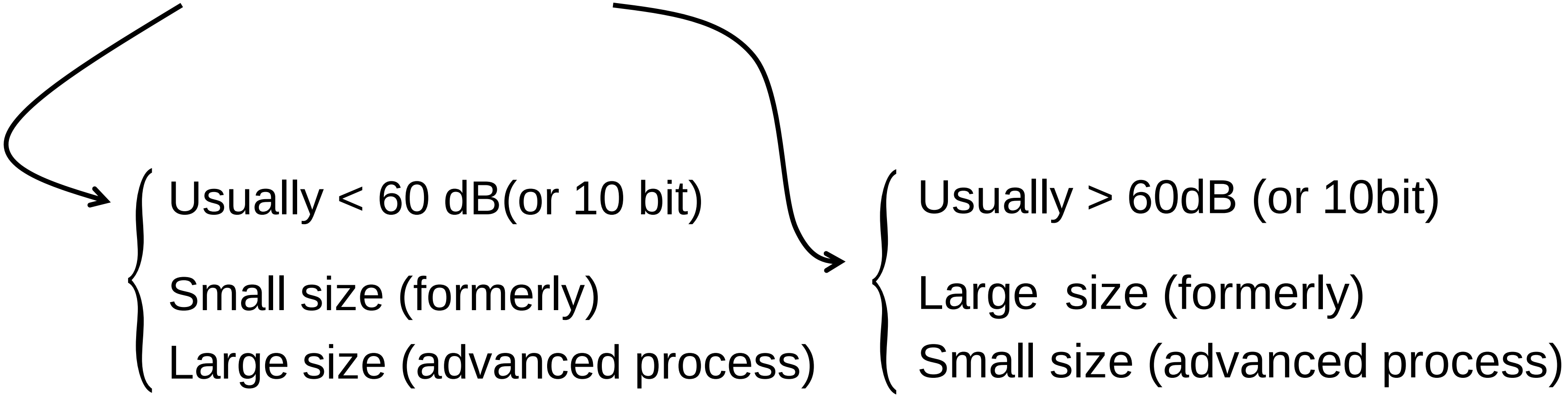
Prefiltering

- Nyquist rate
  - ◆ Prefilter = Anti-alias filter (AAF)
  - ◆ Brick wall AAF
- Oversampling rate
  - ◆ Prefilter = Anti-alias filter + Decimation filter

    (AAF) (DF)

    AAF: Continuous-time filter

    DF: SCF or Digital filter



Usually < 60 dB(or 10 bit)

Small size (formerly)

Large size (advanced process)

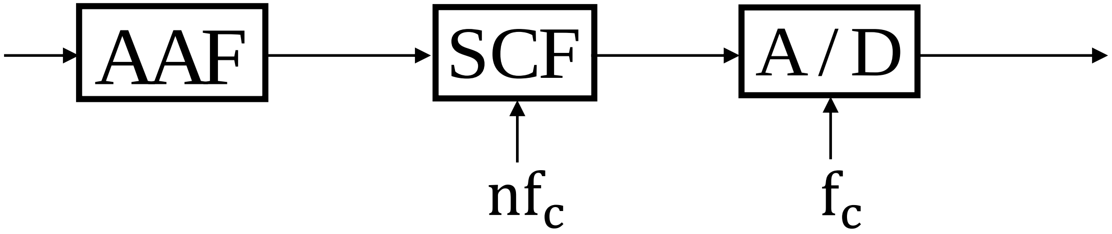
Usually > 60dB (or 10bit)

Large size (formerly)

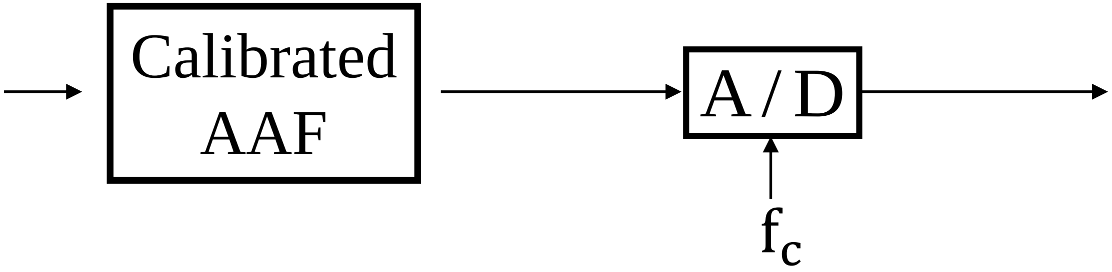
Small size (advanced process)

Prefiltering (Cont.)

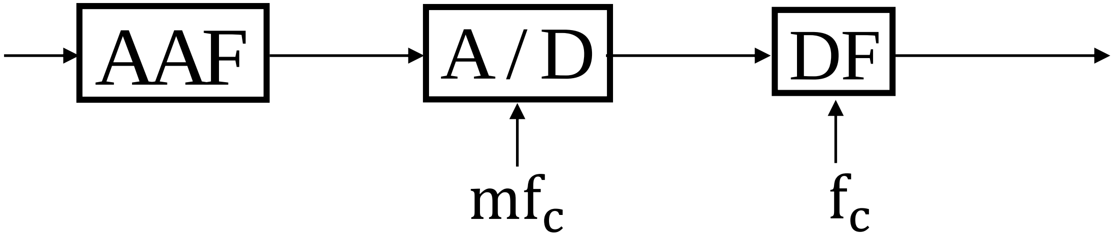
- Examples: (For Data Acquisition)
  - ◆ Conventional Nyquist-rate A/D converter



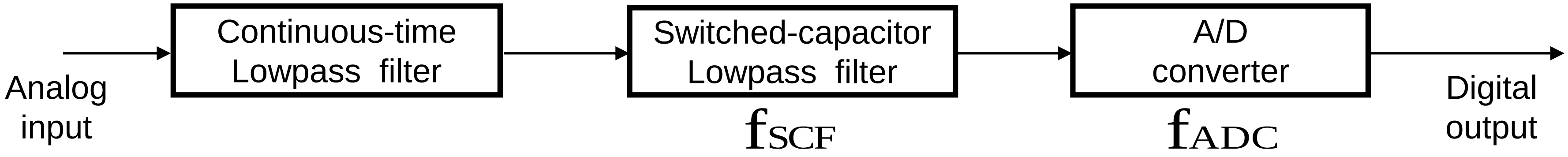
  - ◆ Modern high-speed design with Nyquist-rate A/D converter



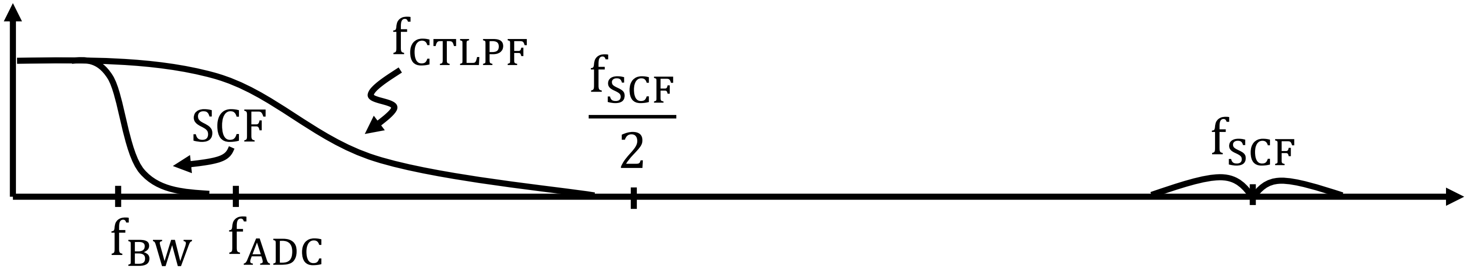
  - ◆ Oversampling A/D converter



Prefilter Strategy for Conventional Data Acquisition

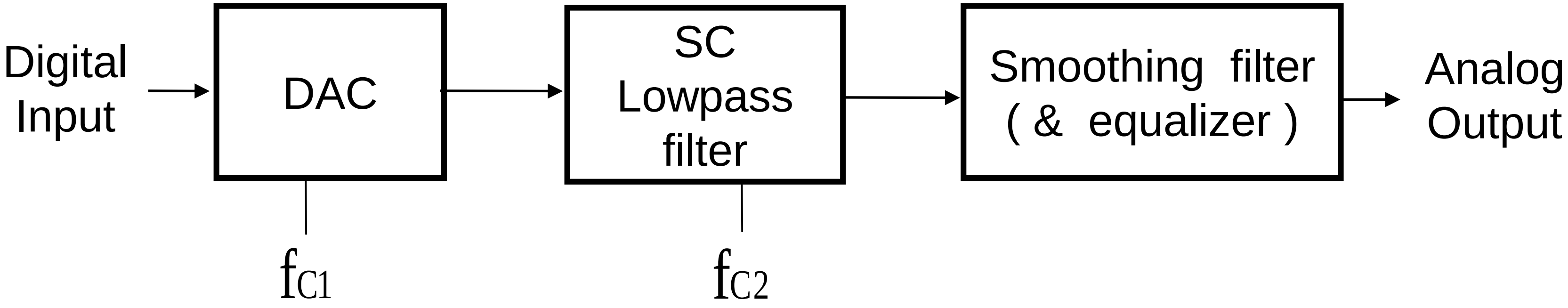


- Provide anti-aliasing for ADC with SCF sampling at a frequency much greater than twice the bandwidth of interest.
- Provide anti-aliasing for SCF by continuous-time low-pass filter (CTLPF) with corner frequency comfortably between ${}^{f_{SCF}}/_{2}$ and f_{BW}

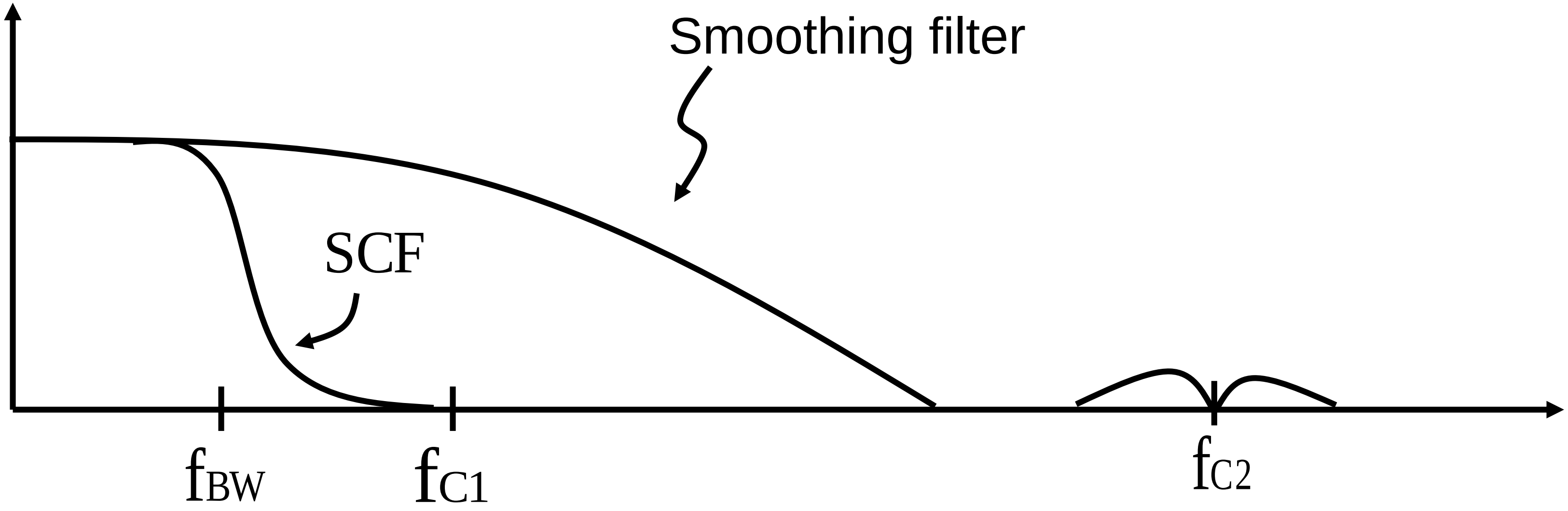


- $f_{SCF} >> f_{ADC}$ → SCF performs anti-aliasing filter → Decimation occurs without aliasing effect → ADC performs decimation and A/D conversion.
- Sometimes, SCF instead of ADC performs filtering and decimation. For either way, control clocks had better be synchronized.

Conventional Postfiltering

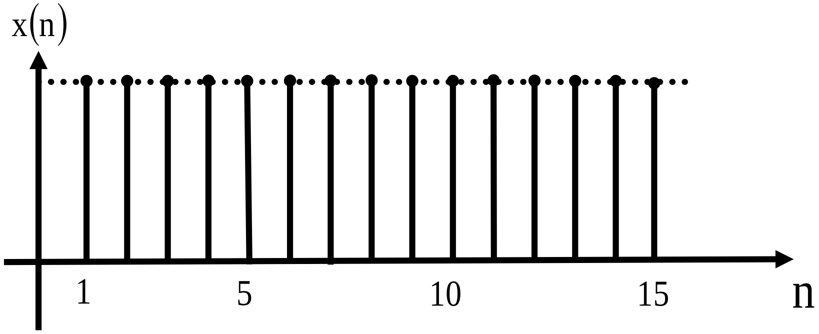


Digital Input → DAC → SC Lowpass filter → Smoothing filter (& equalizer) → Analog Output

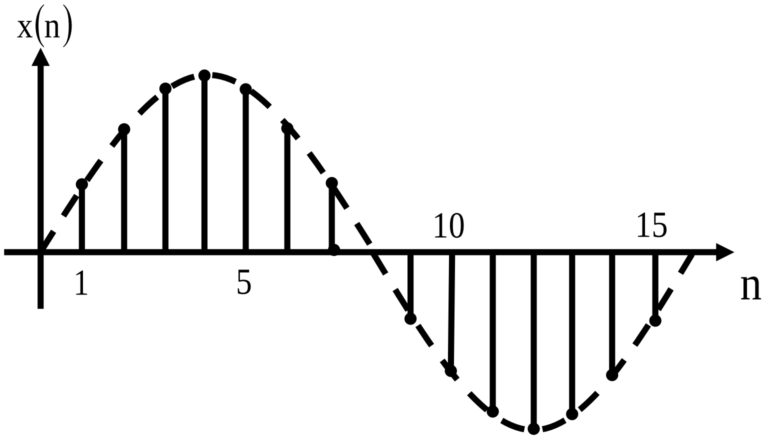


Discrete-time Sinusoidal Signals with Different Sampling Rates

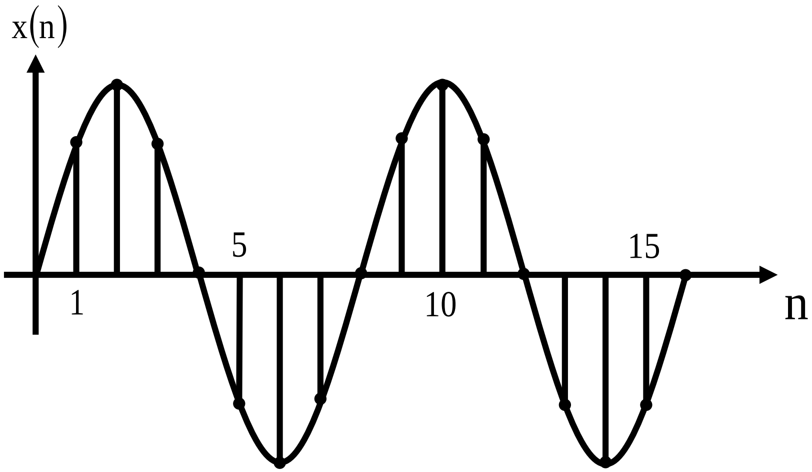
- ∞ samples/cycle



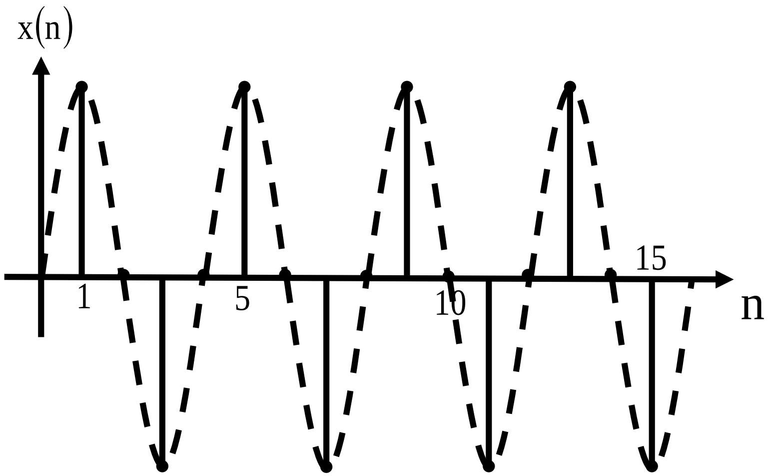
- 16 samples/cycle



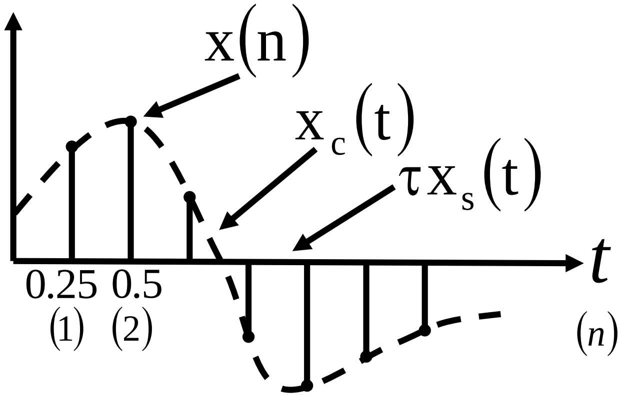
- 8 samples/cycle

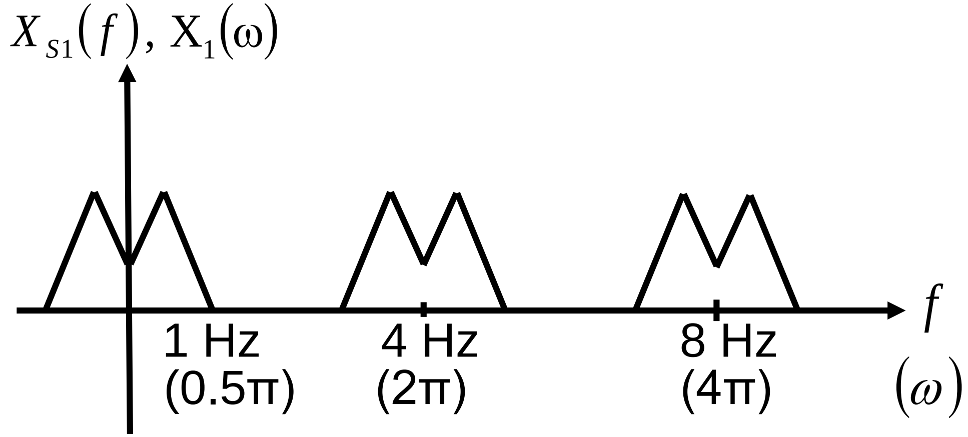


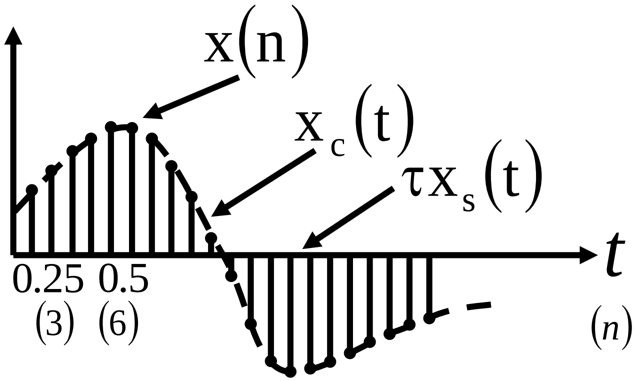
- 4 samples/cycle

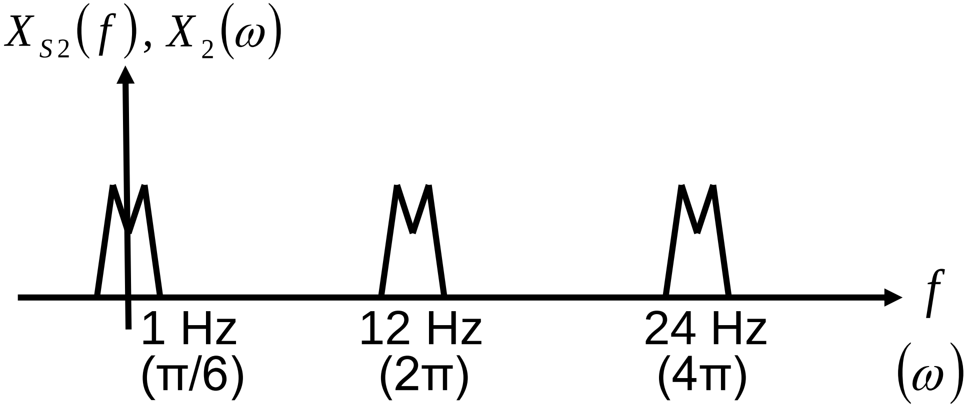


Time and Frequency Domain of Different Sampling Rates









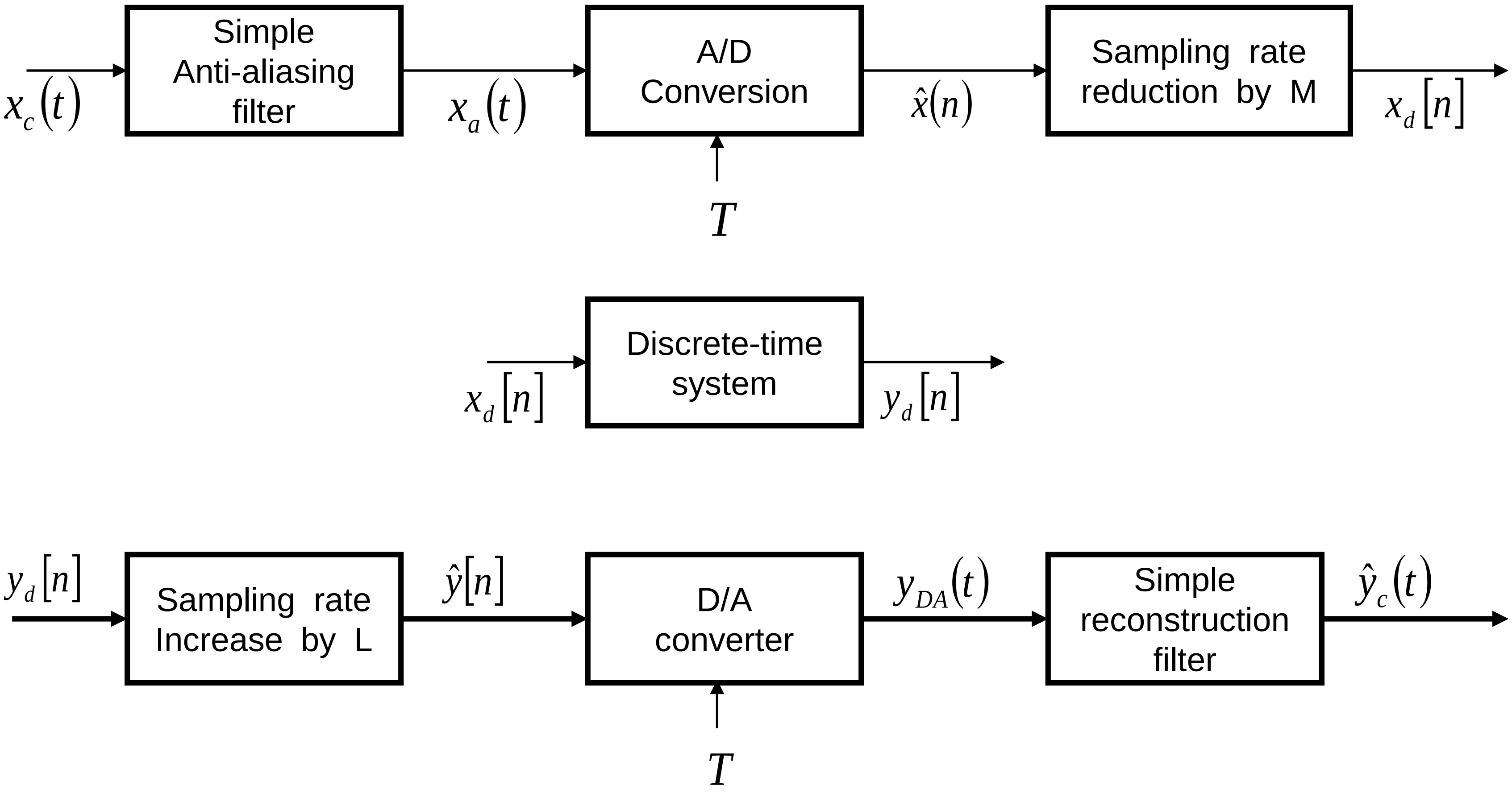
Use of Oversampling Approach to Relax Requirements of Prefilter and Postfilter

- Front End
  - Use oversampling A/D converter
  - Use decimation after A/D conversion

- Back End
  - Use interpolation before D/A conversion
  - Use oversampling D/A converter

Use of Oversampling Approach to Relax Requirements of Prefilter and Postfilter (Cont.)

- Example: Block diagram of a signal processing system



$x_c(t)$

Simple Anti-aliasing filter

$x_a(t)$

A/D Conversion

T

$\hat{x}(n)$

Sampling rate reduction by M

$x_d[n]$

$x_d[n]$

Discrete-time system

$y_d[n]$

$y_d[n]$

Sampling rate Increase by L

$\hat{y}[n]$

D/A converter

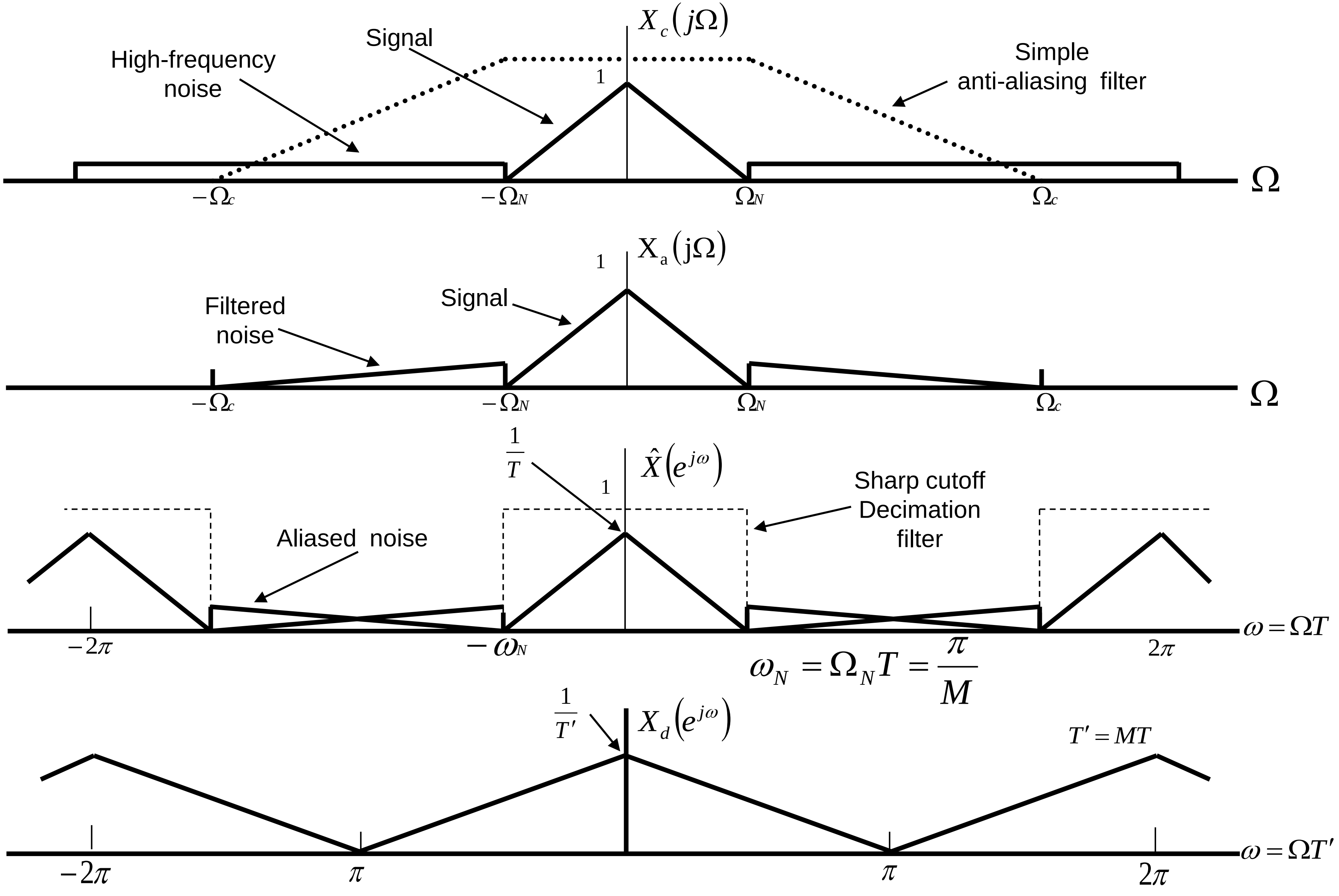
T

$y_{DA}(t)$

Simple reconstruction filter

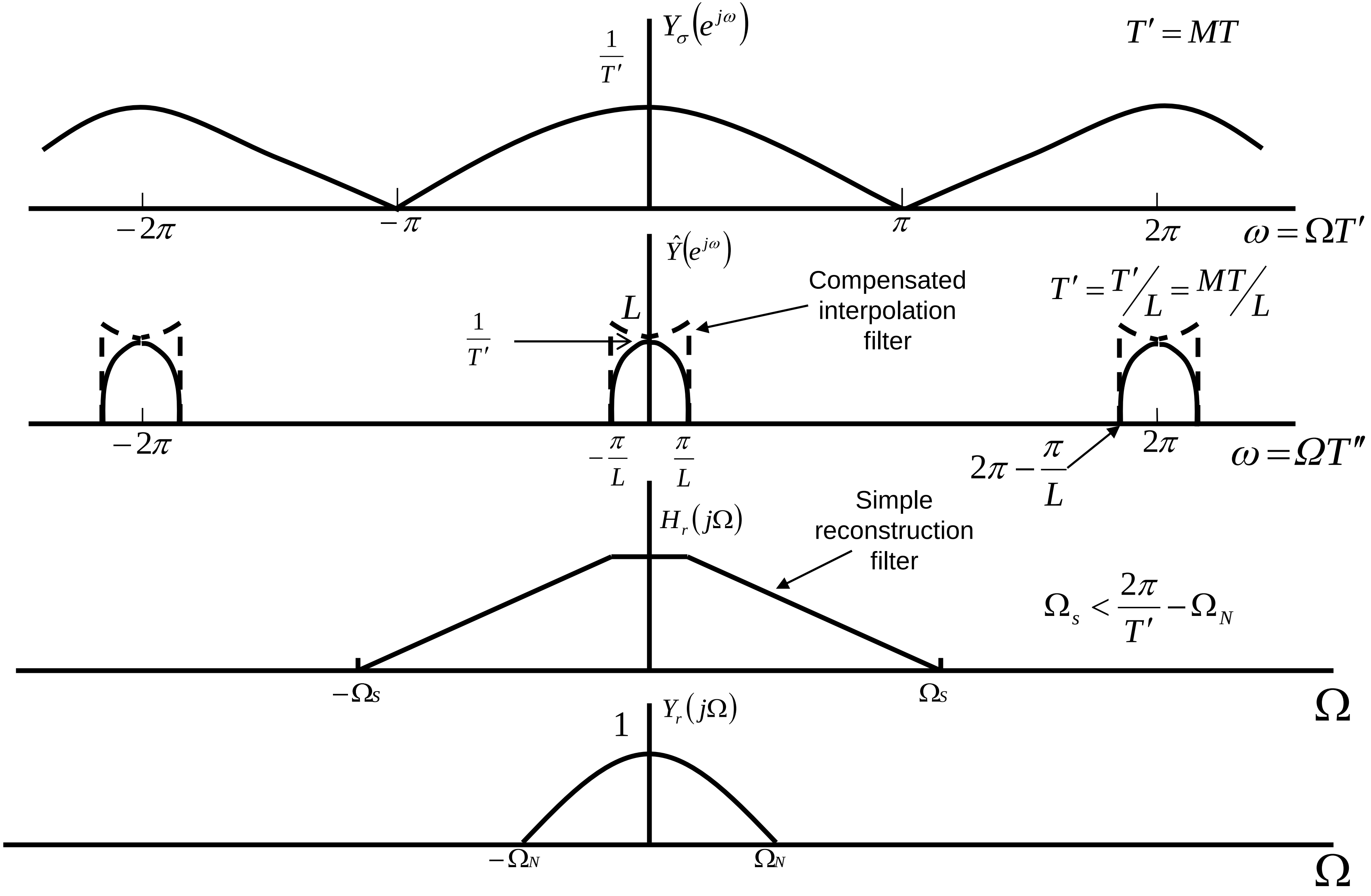
$\hat{y}_c(t)$

Use of Decimation in A/D Conversion



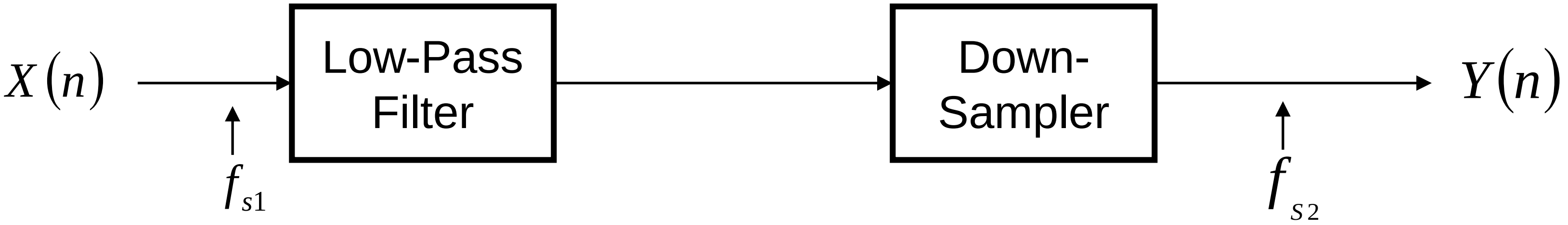
$$\omega_N = \Omega_N T = \frac{\pi}{M}$$

Use of Interpolation in D/A conversion

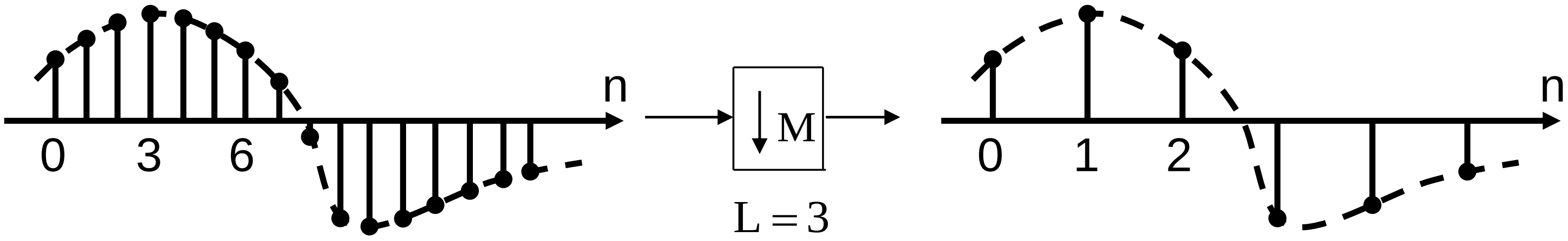


Decimation

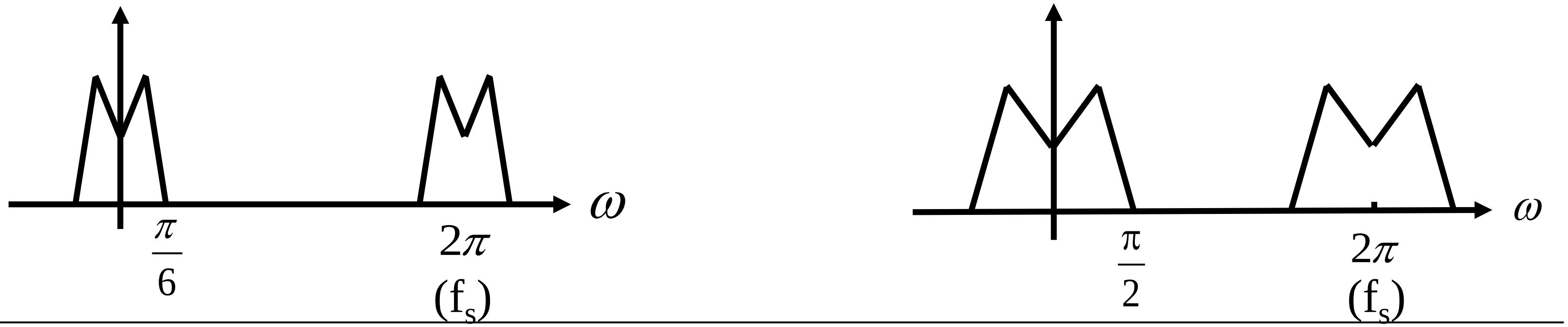
- Lowpass filtering + downsampling



- Downsampling (by 3)
  - Time-domain

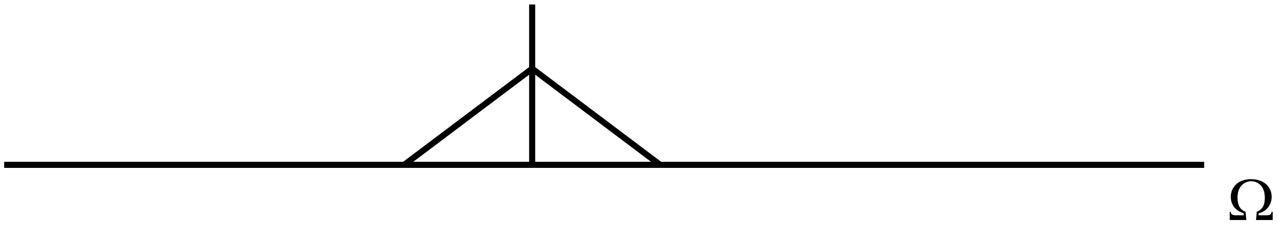


  - Frequency domain

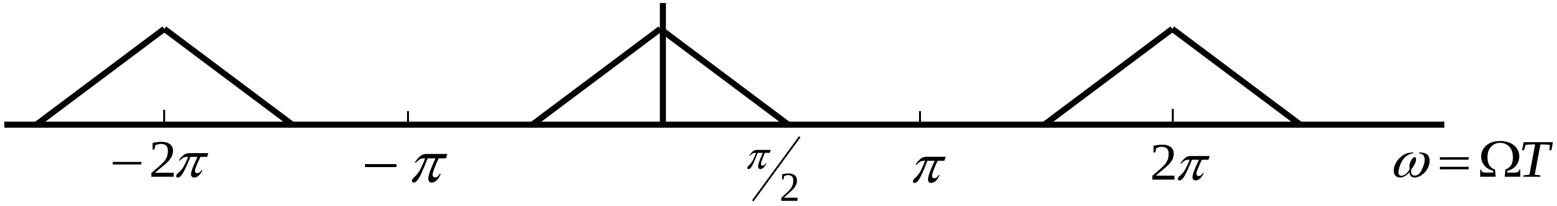


Decimation (Cont.)

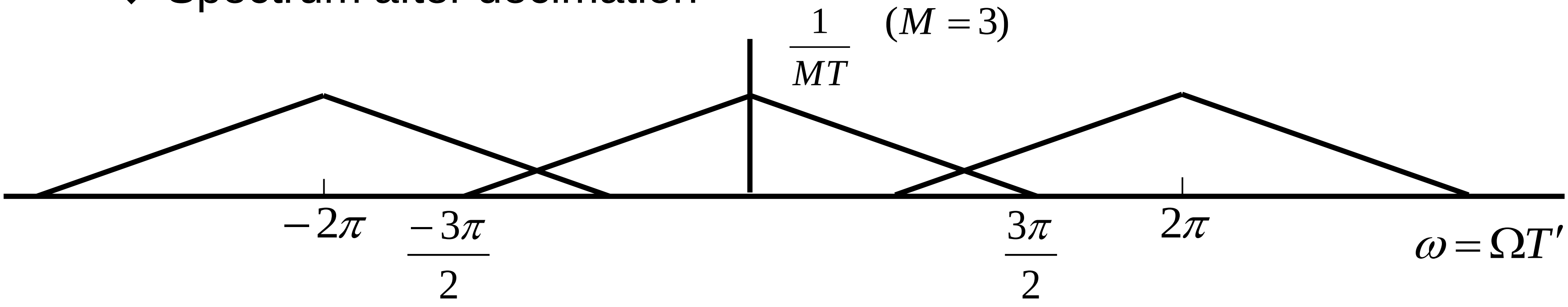
- Downsampling with aliasing
  - ◆ Spectrum of original continuous-time signal



  - ◆ Spectrum of sampled signal

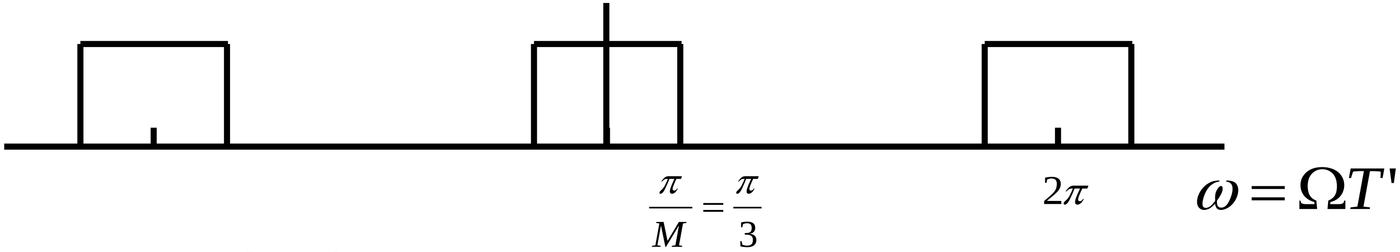


  - ◆ Spectrum after decimation

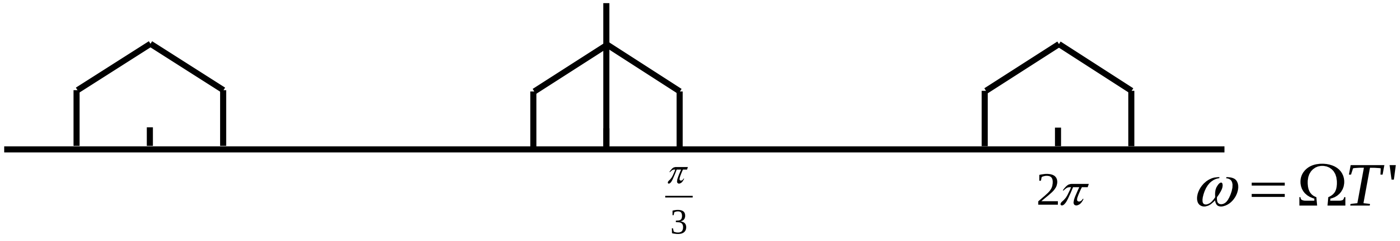


Decimation (Cont.)

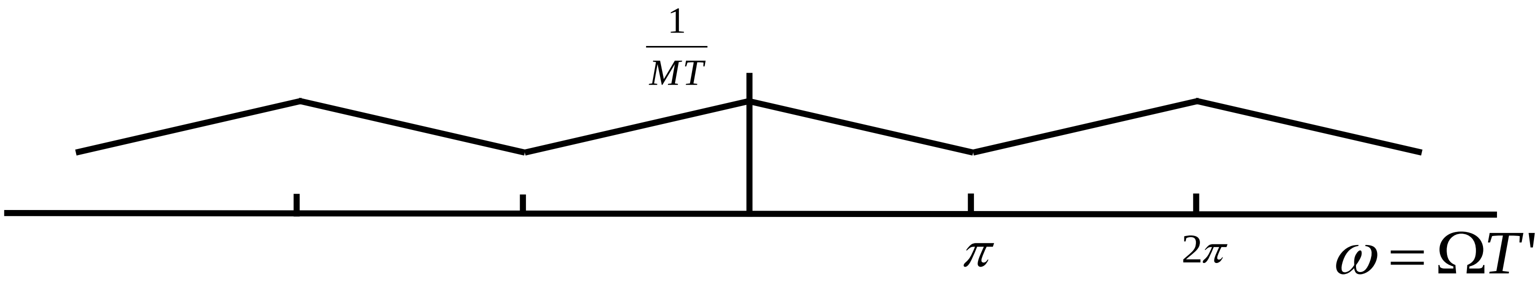
- Downsampling with prefiltering to avoid aliasing
  - ◆ Low-pass filtering



  - ◆ Spectrum after filtering

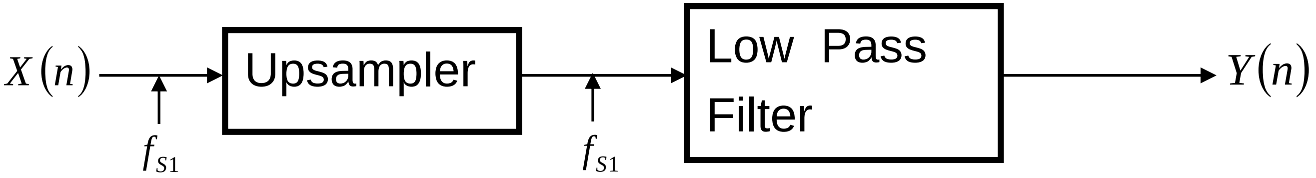


  - ◆ Spectrum after decimation

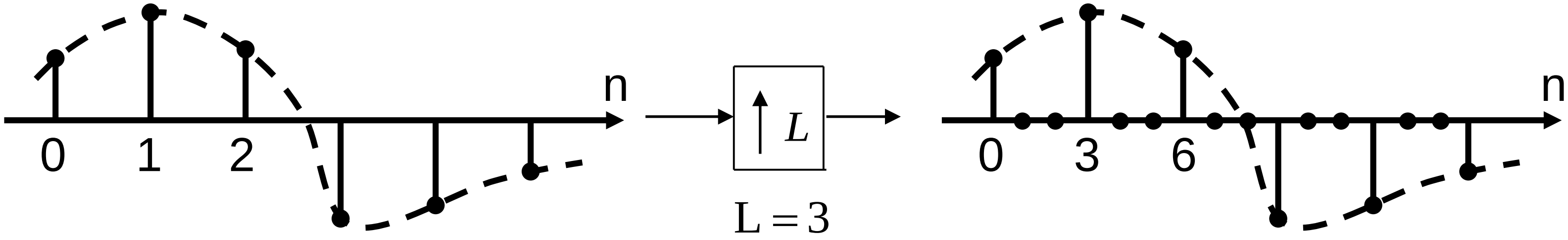


Interpolation

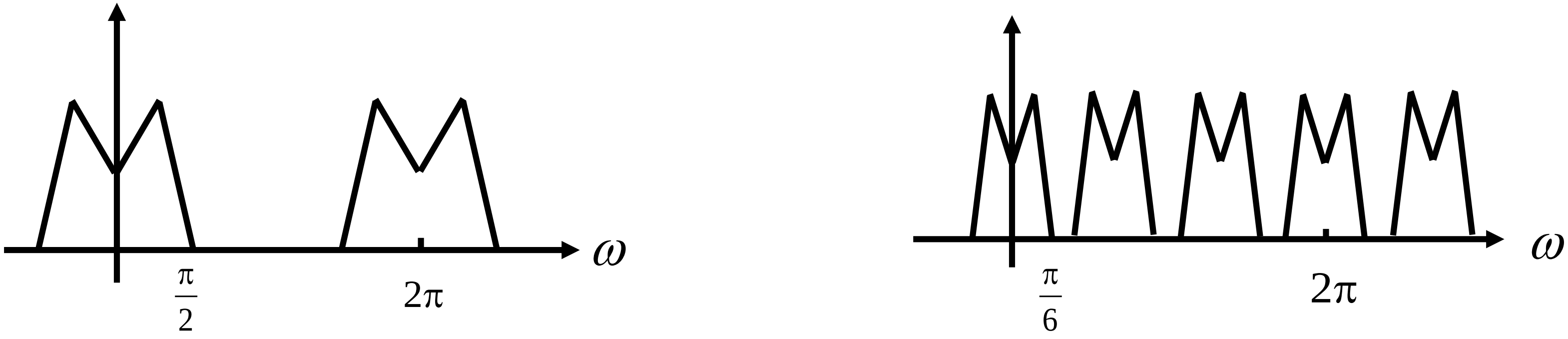
- Upsampling + lowpass filtering
  - ◆ Time-domain



- Upsampling (by 3)
  - ◆ Time-domain

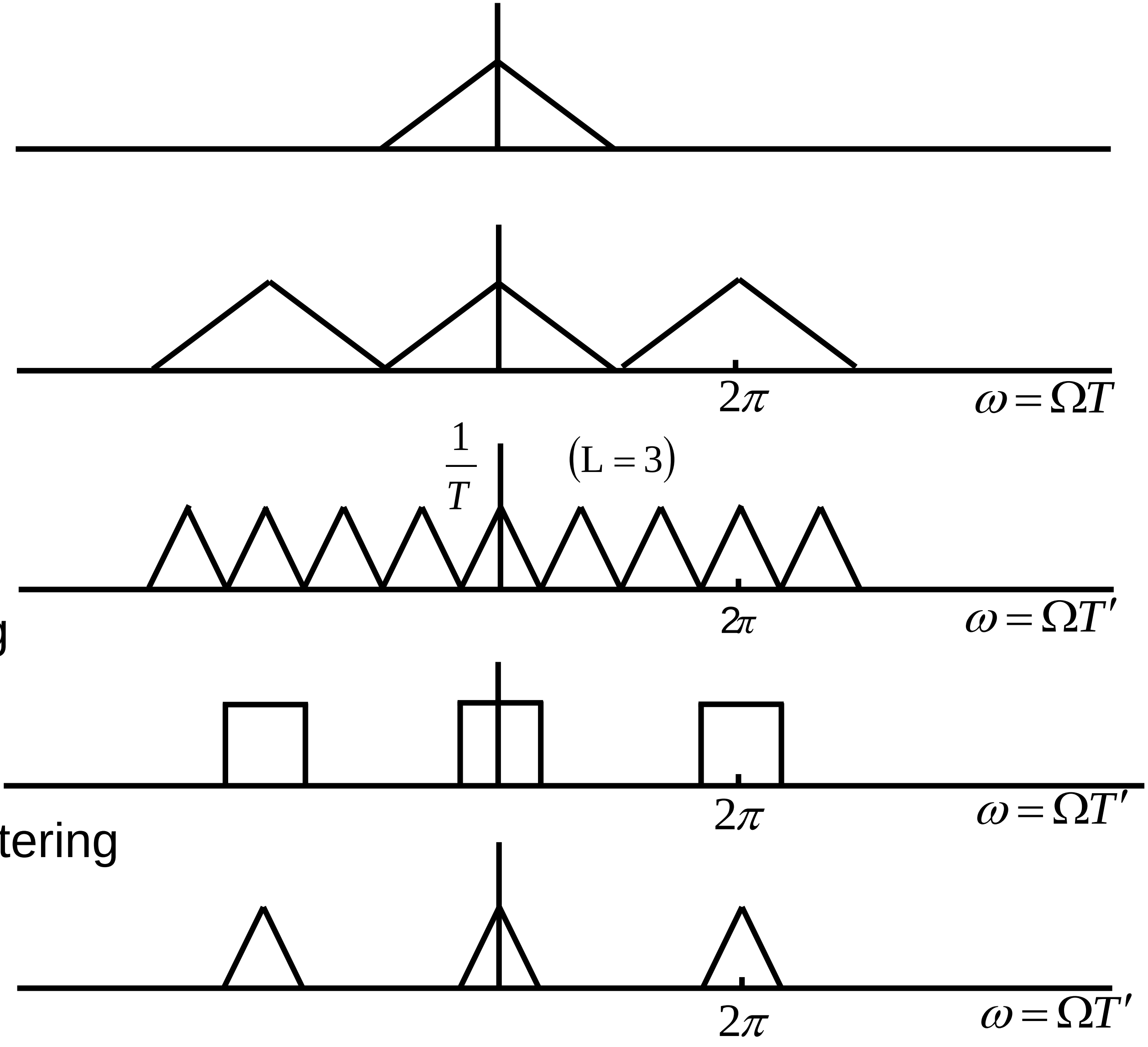


  - ◆ Frequency-domain



Interpolation (Cont.)

- Baseband signal (Analog)
- Original signal (Digital)
- Upsampling
- Low-pass filtering
- Spectrum after filtering



Decimate-By-N Filter

- Example: moving average sinc function

Input rate f_c

Output rate ${}^{f_c}/_N = f_d$

$$Y(n) = \frac{1}{N}[X(Nn) + X(Nn-1) + X(Nn-2) + \cdots + X(Nn-N+1)\ ]$$

➔ $Y(z) = \frac{1}{N}[X(z) + z^{-1}X(z) + \cdots + z^{-(N-1)}X(z)]$

➔ Transfer function $\frac{Y(z)}{X(z)} = \frac{1}{N}[\frac{1-z^{-N}}{1-z^{-1}}]$

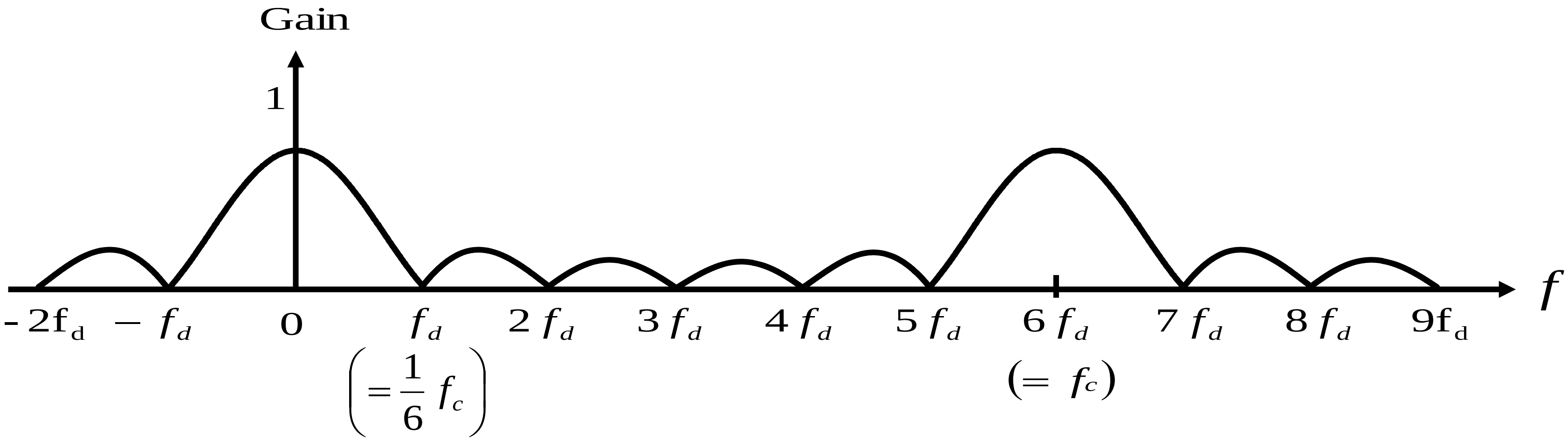
Decimate-By-N Filter (Cont.)

- Gain of sinc filter

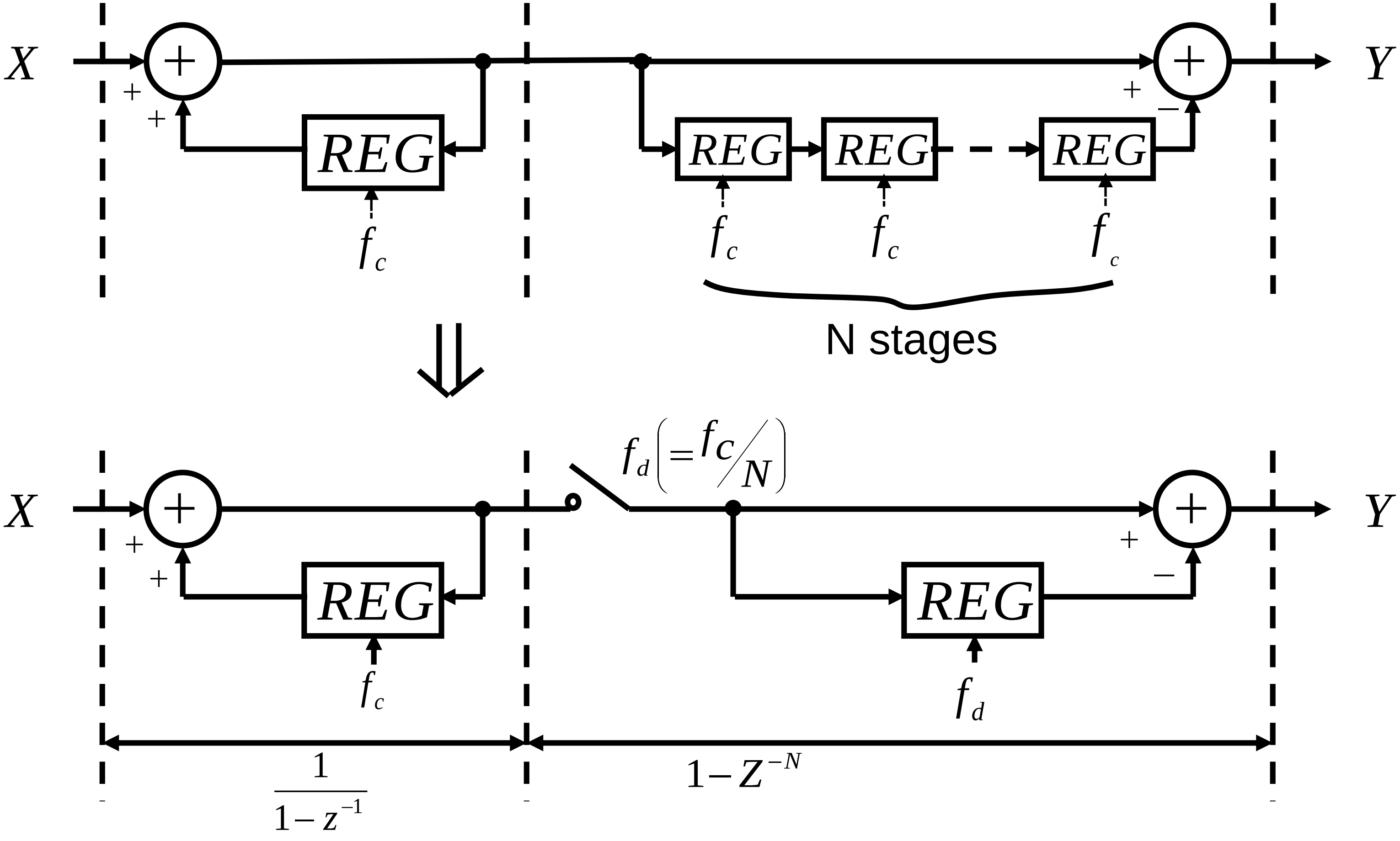
$$\frac{1}{N}\left|\frac{1-z^{-N}}{1-z^{-1}}\right| = \frac{1}{N}\left|\frac{\sin\frac{N\omega\tau}{2}}{\sin\frac{\omega\tau}{2}}\right| \quad ; \quad z = e^{j\omega\tau} \quad \text{and} \quad \tau = \frac{1}{f_c}$$

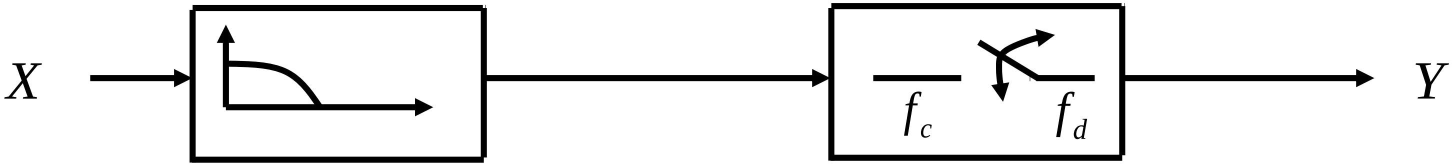
$$= \left|\frac{\operatorname{sinc}\frac{N\omega\tau}{2}}{\operatorname{sinc}\frac{\omega\tau}{2}}\right| \quad ; \quad \operatorname{sinc}X = \frac{\sin X}{X}$$

- Example: N=6



Digital Decimator with Sinc Filtering





Lowpass Filter

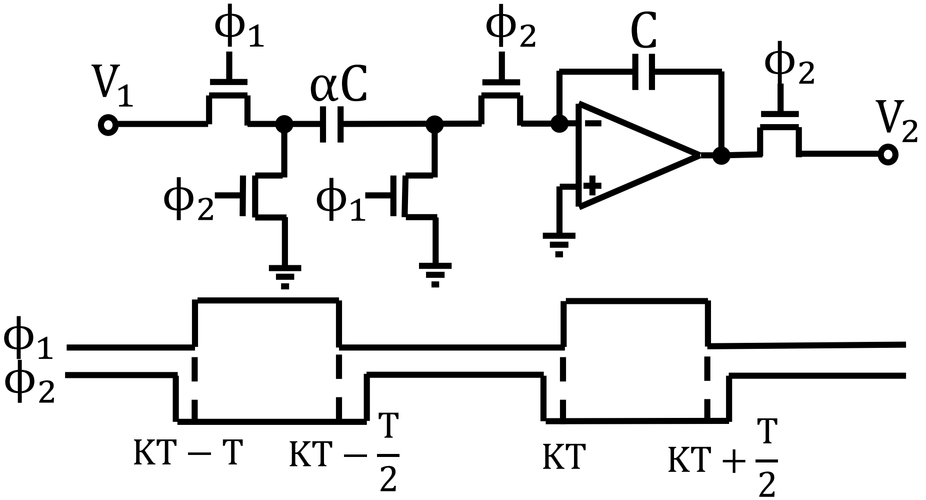
Decimator

SC Sampling Stage Without Decimation

- Example: Integrator

$$V_2(kT) - V_2(kT-T) = \alpha V_1\left(kT - \frac{T}{2}\right)$$

$$H_I(z) = \frac{V_2(z)}{V_1(z)} = \frac{\alpha z^{-\frac{1}{2}}}{1-z^{-1}}$$

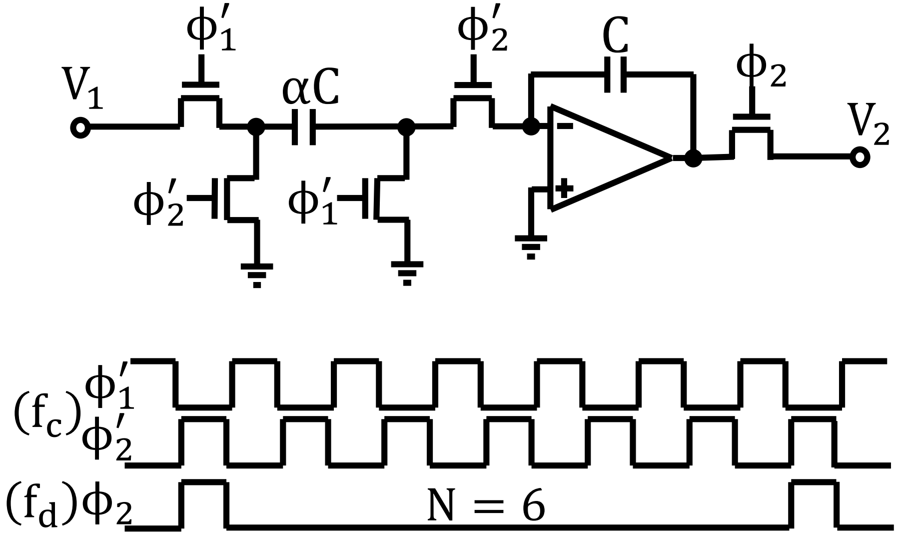


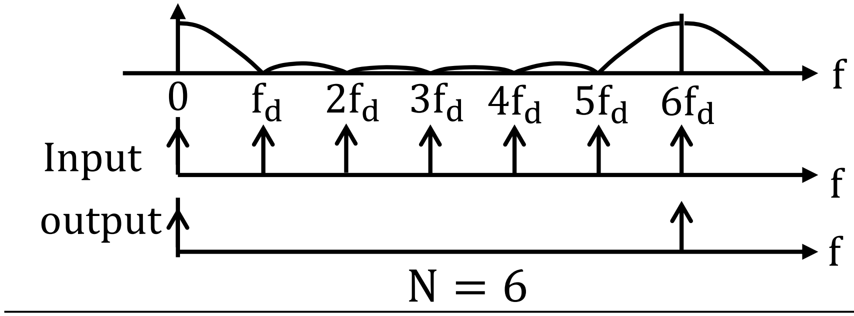
- Example: Modified Integrator

$$H(z) = H_I(z)H_D(z)$$

$H_D(z)$: sinc transfer function

$$\left|H_D\left(e^{j\omega\tau}\right)\right| = \left|\frac{\sin\frac{N\omega\tau}{2}}{\frac{\omega\tau}{2}}\right| \; ; \; \tau = \frac{1}{f_c} = T$$

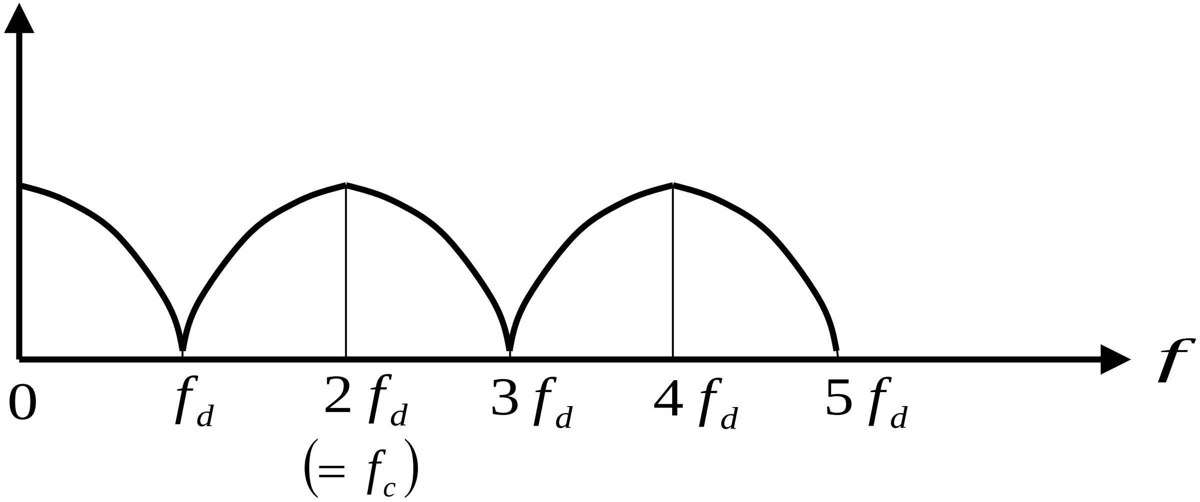




Cosine Filter

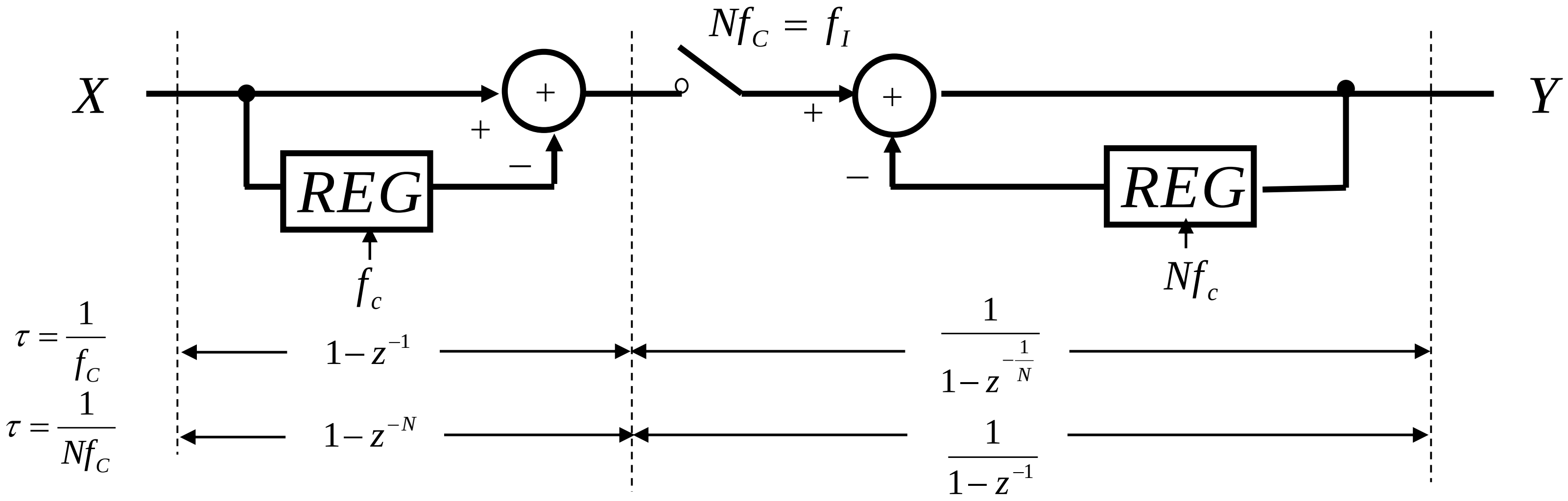
- For N=2

$$\left|H_I\left(e^{j\omega\tau}\right)\right| = \left|\frac{\sin\omega\tau}{\sin\frac{\omega\tau}{2}}\right| \quad ; \quad \tau = \frac{1}{f_c}$$

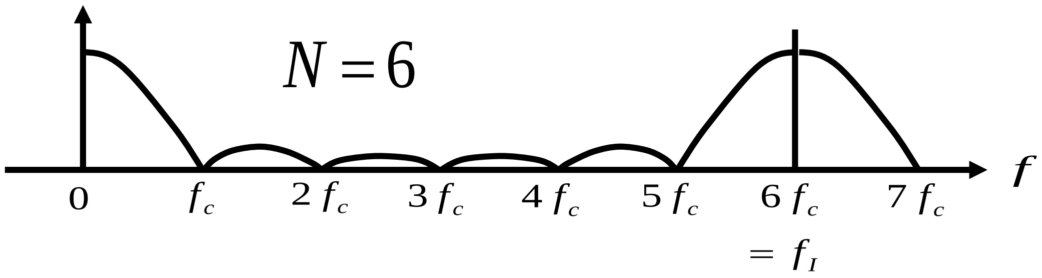


Interpolation Filter

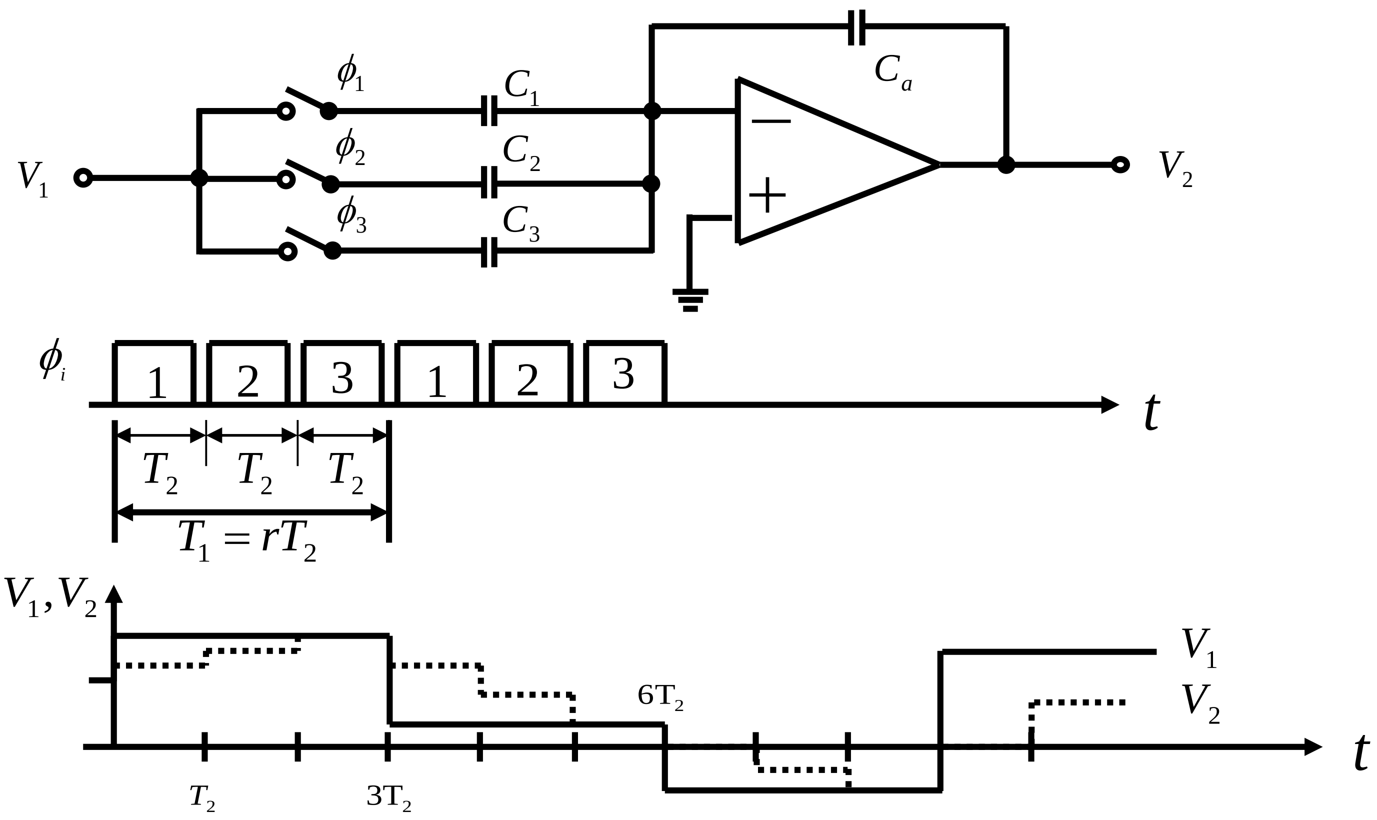
- Example: Linear Interpolation (sinc function)



$$H_I(z) = \frac{1-z^{-N}}{1-z^{-1}} \quad ; \quad H_I(z) = \frac{1}{N}\left|\frac{\sin\frac{N\omega\tau}{2}}{\sin\frac{\omega\tau}{2}}\right| \quad ; \quad \tau = \frac{1}{f_I}$$



SC Sampling Stage With Linear Interpolation



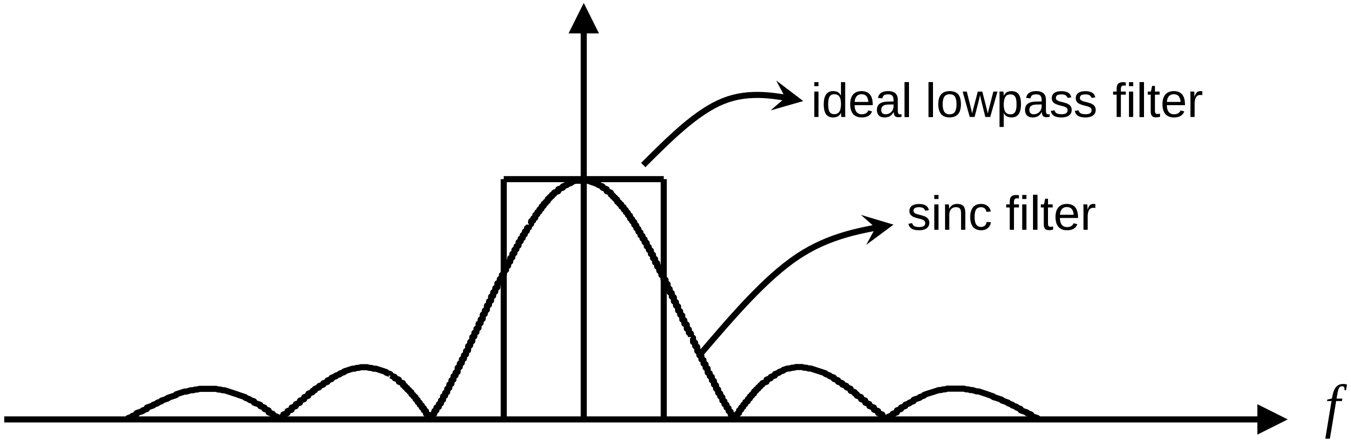
$$V_2(nT_2)-V_2(nT_2-T_2)=\frac{1}{N}\left[V_1(nT_2)-V_1(nT_2-NT_2)\right]$$

$$H_I(z)=\frac{V_2(z)}{V_1(z)}=\frac{1}{N}\frac{1-z^{-N}}{1-z^{-1}}=\frac{1}{N}\sum_{i=0}^{N-1}z^{-i}$$

$$\left|H_I\left(e^{j\omega\tau}\right)\right|=\frac{1}{N}\left|\frac{\sin\frac{N\omega\tau}{2}}{\sin\frac{\omega\tau}{2}}\right| \quad ; \quad \tau=\frac{1}{f}=T_2$$

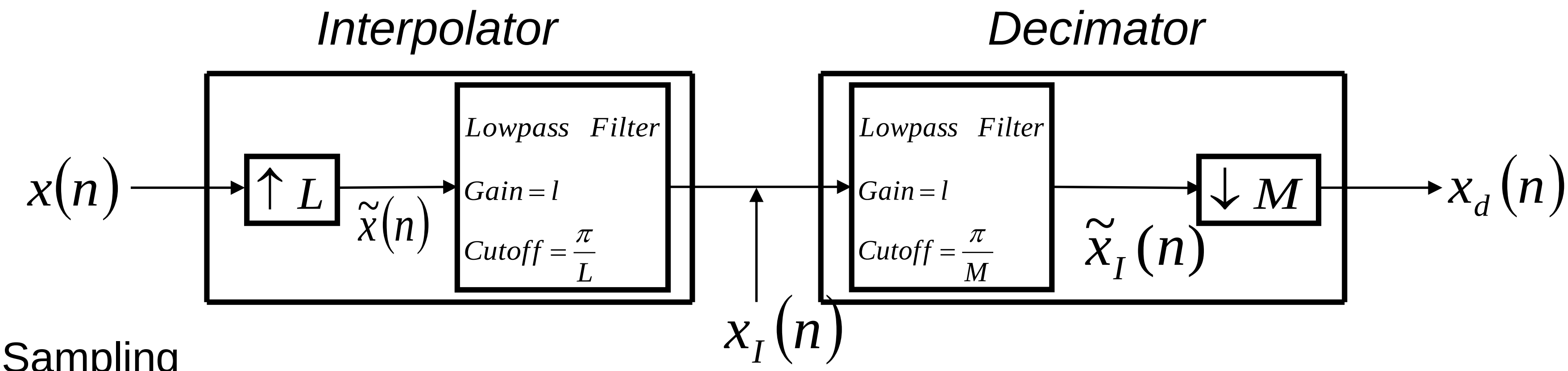
Sinc Filtering Function

- Not ideal lowpass
- FIR (Finite Impulse Response)



- To implement ideal lowpass function, other approaches can be used.

Changing The Sampling Rate By A Noninteger Factor



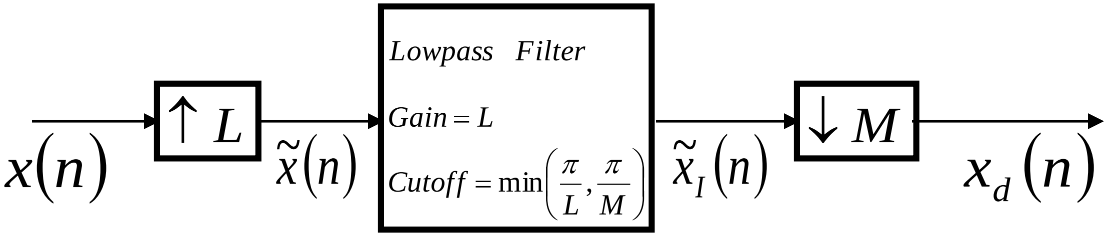
Interpolator

Decimator

$x(n)$ → $\uparrow L$ → $\tilde{x}(n)$ → Lowpass Filter, $Gain = l$, $Cutoff = \frac{\pi}{L}$ → $x_I(n)$ → Lowpass Filter, $Gain = l$, $Cutoff = \frac{\pi}{M}$ → $\tilde{x}_I(n)$ → $\downarrow M$ → $x_d(n)$

Sampling Period :T T/L T/L T/L TM/L

⇩ Two lowpass filters can be combined



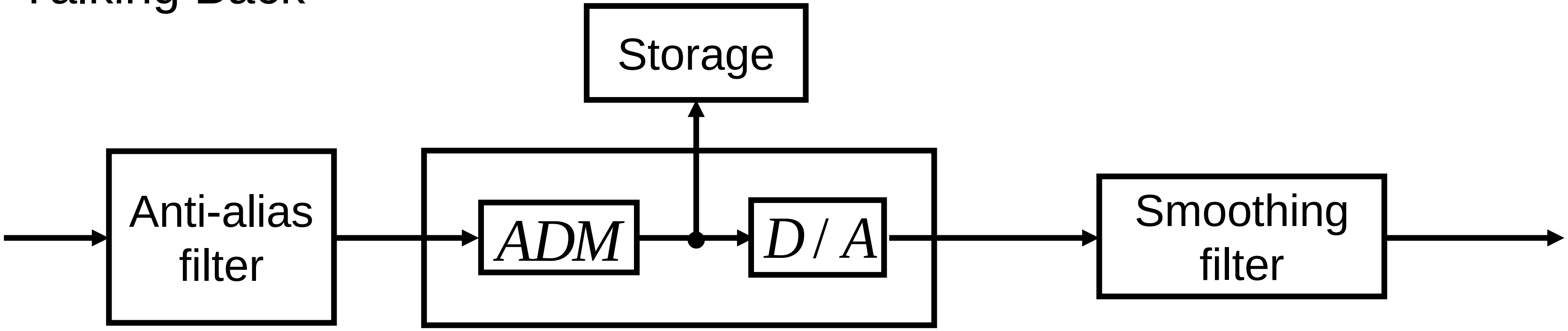
$x(n)$ → $\uparrow L$ → $\tilde{x}(n)$ → Lowpass Filter, $Gain = L$, $Cutoff = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$ → $\tilde{x}_I(n)$ → $\downarrow M$ → $x_d(n)$

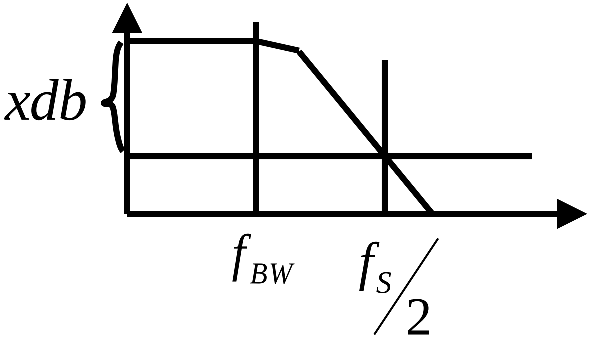
Sampling Period : T T/L T/L TM/L

- Noninteger factors can be obtained from properly choosing M and L .

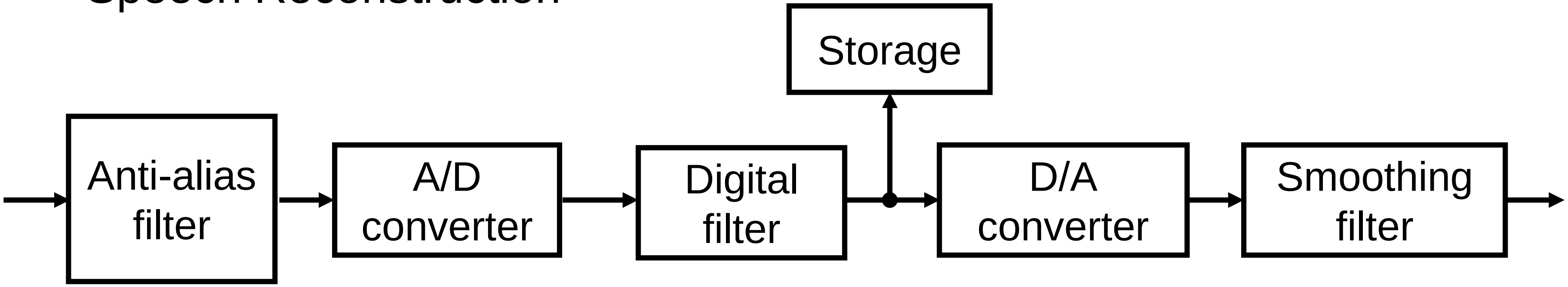
Examples

- Talking Back





- Speech Reconstruction



Examples (Cont.)

- Approach 1:

  continuous-time + oversampling + $\overbrace{\text{digital}}^{\text{Better LPF can be used other than sinc one}}$ + DAC + SMF
  AAF ADC Decimator
  and LPF

  - Oversampling ADC and digital decimator may be combined.

- Approach 2:

  continuous-time + SCF + conventional + DAC + SMF
  AAF ADC
  (Nyquist rate)